AN EFFICIENT GALERKIN METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO DARCY'S EQUATION

BY

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Dedication

To my parents, wife, children and to brothers and sisters

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First and foremost to ALLAH who gave me the courage and patience to carry out this work. Peace and blessings of ALLAH be upon his Last messenger Mohammed (SAW), the guide of humanity.

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DISSERTATION ABSTRACT

NAME:	Radwan Ali Ali Al-Rubaee
TITLE OF STUDY:	**An Efficient Galerkin Method for Stochastic Differential Equations with Applications to Darcy's Equation**
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In this dissertation, we study a stochastic version of Darcy's equation which arises the coefficients are stochastic as well as right hand side. We present a new basis for a subspace of martingales in which we expand the stochastic coefficients. This enable us to transform the stochastic partial differential equation into a sequence of deterministic partial differential equations. Discretization is performed by stochastic Gelerkin finite element method which combines mixed finite element in the computational domain with basis of random functions to handle the random part. Issues of existence, uniqueness, stability and order of convergence are investigated. Finally, we experiment with real permeability data taken from sandstone core.

ملخــص الرســـــالة

الاسم : رضوان علي علي الرباعي

عنوان الرسالة : طريقة جالركن فعالة للمعادلات التفاضلية بمتغيرات عشوائية مع تطبيقاتها لمعادلة دارسي.

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في هذه الرسالة ، سوف ندرس معادلة دارسي بمعاملات عشوائية وطرف أيمن عشوائي. سوف ننشئ أساساً جزئياً جديداً مكوناً من دوال مرتنجيل والتي تستخدم لنشر المعاملات العشوائية. هذا الأساس يمكننا من تحويل المعادلات التفاضلية الجزئيه العشوائية إلي سلسلة من المعادلات التفاضلية الجزئية العادية. سنستخدم طريقة جالركن للعناصر المحدودة لتقطيع المسأله من خلال جمع العناصر المختلطة والمحدودة في الفضاء الحسابي مع أساس من الدوال العشوائية في الجزء العشوائي. في هذه الدراسة سيتم التحقق من قضايا الوجود والتفرد والأستقرار ورتبه التقارب . أخيرأ سوف نختبر المعادلة باستخدام بيانات حقيقية للنفاذية مأخوذة من الحجر الرملي.

CHAPTER 1

INTRODUCTION

This dissertation aims to study the approximation of solutions for a category of stochastic partial differential equations with application to Darcy's Equation. We combine the spatial space and the stochastic part using a Galerkin approach to approximate solution of the original problem. This chapter presents motivation and arguments for the significance of this research. The motivation of the current research will be explained in Section 1.1. In Section 1.2 we state the problem. In Section 1.3 we describe the research objectives. Review of the literature is discussed in Section 1.4. In Section 1.5 we introduce our approach to achieve the research objectives. We give an outline of the remaining chapters in Section 1.6.

1.1 Motivation

In many practical situations, deterministic functions do not represent a real life model. A suitable model is then that is based upon random variables. The governing equations in such models are stochastic partial differential equations (SPDE) which contain random data. This is an exciting field which brings together ideas of probability theory, functional analysis and the theory of partial differential equations. There are several ways to approximate solutions of the SPDE's. One particular way is to regard it as a random field. In our model, the SPDE describes the flow of a fluid through a random porous medium which obeys Darcy's Law and related to the so-called saddle point problem. Many authors studied the problem via several approaches. Some used the stochastic Galerkin method [1, 10, 18, 19, 31, 50, 51]. In this method they seek to approximate the solution in some finite dimensional subspace by considering the weak formulation of the original problem. Others used the stochastic collocation method [5, 2, 28], the main idea of this approach is to construct an interpolation function for the unknown stochastic solution using the values at a predetermined set of points in the stochastic direction. These points are called the collocation points.

Generalized polynomial expansion [19] is used to find solutions which are regarded as random processes by using the representation in terms of orthogonal polynomials in the stochastic space. Monte Carlo and quasi Monte carlo Methods [70] are also used for sampling the stochastic term. Indeed, it transforms the stochastic partial differential equation into a deterministic one through this sampling.

1.2 Problem Statement

In this dissertation we will be concerned with a new approach and, as an application, we will consider the boundary value problem.

$$\begin{aligned} A^{-1}(x,\omega)u(x,\omega) + \nabla p(x,\omega) &= 0, && \text{in } D \\ \nabla \cdot u(x,\omega) &= -f(x,\omega) && \text{in } D \\ p(x,\omega) &= g(x,\omega) && \text{on } \partial D_{Dir} \\ n \cdot u(x,\omega) &= 0 && \text{on } \partial D_{Neu} \end{aligned} \tag{1.1}$$

with stochastic boundary condition and $(\Omega, \mathcal{F}, P)$ a complete probability space, where Ω is the trial space, $\mathcal{F}$ a special σ-algebra of subsets of Ω, $P : \mathcal{F} \to [0,1]$ is a probability measure, and $A^{-1}(x,\omega)$ is a random field, i.e, a random variable at every $x \in D$. The solution to the problem consists of two random fields $(u,p) = (u(x,\omega), p(x,\omega))$.

1.3 Research Objectives

We intend to study the boundary value problem (1.1) with a new approach. We will focus on the following :

(i) Introduce the Darcy equation with stochastic coefficients.

(ii) Introduce a method to transform the Darcy equation with stochastic coefficients to one with deterministic coefficients.

(iii) Study the questions of existence, uniqueness and approximation properties.

(iv) Study numerical methods for solving the transformed equations, their accuracy and their preconditioning.

(v) Experiment with real permeability data from sandstone core.

1.4 Literature Review

The reservoir simulation has been studied through the numerical approximations of the elliptic PDE involved [7, 12, 21, 22, 28, 31, 32, 33, 39, 56]. The corresponding problem with random data is still an active area of research [23, 26].

Several authors have considered numerical methods for the problem (2.1). For example, Powell, C.E. et. al [50] introduced a stochastic Galerkin mixed formulation in which a standard finite element discretization with two different types of stochastic basis functions are introduced and used with the minimum residual method. Ganis, B. et.al [5], used mixed finite element approximation in the spatial domain and collocation at zero as tensor product of hermit polynomials in the stochastic direction.

Xiu, D. et.al, [19] transformed the problem into a set of deterministic equations using generalized polynomial expansion as to write the solution as a convergent series.

Babuska, I. et. al [28], considered the input data depending upon finite number of random variables and used mixed Galerkin approximation in space and collocation method in the stochastic direction using orthogonal polynomial.

Elman, H. C., et. al [26], considered the steady-state diffusion problem with random data using $H(div)$ preconditioner for the resulting saddle point systems.

Powell, C.E. et. al [10] used mixed finite element discretization in space with global polynomial approximation in probability space.

Besapalove, A. et. al, [1] studied a first-order system of PDEs with random coefficients under the small noise assumption. They combine mixed finite elements in space with M-variate tensor product polynomials in the stochastic direction.

Andrew D. et.al [2] used stochastic collocation methods where elliptic PDEs with random diffusion coefficients are discretized using mixed finite element method in the space to obtain a saddle point formulation and use Raviart-Thomas elements.

Frances, et. al [70] used the quasi-Monte carlo method to approximate expected values of the linear functionals of the solution of the PDE by considering these as infinite dimensional integrals in the parameter space.

Silvester D. et. al [18] used the orthogonal polynomials in the stochastic domain while the space inf-sup stable Taylor Hood approximate in space to get non symmetric saddle point problem.

1.5 Research Approach

In this dissertation we start by constructing a basis for a subspace of $L^2(\Omega, \mathcal{F}, P)$, which can be used to approximate the stochastic part of the coefficients solution of the differential equation. The result will be in a sequence of deterministic partial differential equations. Then we solve these deterministic equations using Raviart-

Thomas finite elements. This type of finite element method requires solving a huge linear system with very large condition numbers. We solve this problem by introducing a block tridiagonal preconditioner which has a good eigenvalue clustering behavior.

1.6 Dissertation Outline

The dissertation is divided into seven chapters, including the introduction chapter briefly introduces the research, provides motivation, states the problem, the research objectives, literature review and the research approach.

Chapter 2 presents a brief description of mathematical preliminaries, random variables and its spaces, random fields, Karhunen-Loeve Expansion and stochastic Galerkin Method.

Chapter 3 introduce an orthonormal basis of martingales. This space is a subspaces of $L^2(\Omega, \mathcal{F}, P)$ which is edequate for the class of Darcy problem considered here.

Chapter 4 focuses on theoretical aspects of the problem. The saddle point problem and a perturbed saddle point problem with random data are discussed. This chapter present expression for the random fields. The Galekin approximation and error analysis is also investigated in this chapter.

Chapter 5 presents block triangular H(div) preconditioners with two parameters which are applicable for the indefinite linear system. A new bounds of the clustering of the eigenvalues are derived for the preconditioned system. Different problems with different types of the permeability coefficient are solved with numerical experiments.

Chapter 6 presents the numerical experiment and approximate solution of the stochastic Darcy Equation (1.1) with stochastic boundary conditions by using the stochastic Galerkin Method. In this chapter we use real permeability data taken from sandstone core.

Chapter 7 discusses the conclusion and future work.

CHAPTER 2

PRELIMINARIES

2.1 Mathematical Preliminaries

In this Chapter we summarize some basic notions from probability theory and stochastic analysis. firstly, we give some useful definitions in probability theory. A detailed accounts can be existed in [43]. A probability space $(\Omega, \mathcal{F}, P)$, consists of Ω, a set of elementary events, $\mathcal{F}$ is a σ-algebra of subsets of Ω and a probability measure P [48]. An event in Ω is denoted by ω so that $\omega \in \Omega$.

Random variables X (RVs) are defined to be measurable functions $X : \Omega \longrightarrow \mathbb{R}$. Thus X has values in $\mathbb{R}$ and induces a probability measure with values in $\mathbb{R}$ induces a probability measure P_X called the probability distribution of X. It is denoted by $F_X(x) = P_X(-\infty, x) = P(X < x)$, and if the probability density exists, then $P_X(x) = \frac{dF_X(x)}{dx}$.

Random variables are usually characterized by their statistics moments defined as expectation

$$E[f(x)] = \int_\Omega f(x) dP(\omega) = \int_\mathbb{R} f(x) dF_X(x), \tag{2.1}$$

where f is function. Now, we define some statistics moments which are the mean $\mu_X = E(X)$, the variance $\mathrm{Var}_X = E[(X - \mu_X)^2]$, the standard deviation $\sigma_X = \sqrt{\mathrm{Var}_X}$. The covariance is a bivariate statistics $\mathrm{cov}(X_1, X_2) = E[(X_1 - \mu_{X_1})(X_2 - \mu_{X_2})]$ of two random variables X_1 and X_2.

In this dissertation, we will deal with continuous random variables. For a single-valued random variable X, the set of values of X for all $\omega \in \Omega$ is called the image $X(\omega)$ of Ω, i.e.

$$\Gamma_X = \{X(\omega) : \omega \in \Omega\} \subseteq \mathbb{R} \tag{2.2}$$

That is, Γ_X is actually the range (of all values) of X on the real line or subset of the real line and therefore it is sometimes also called the state space of X.

2.2 Spaces of Random Variables

We mention some basic notations and results in stochastic analysis that are relevant to the stochastic partial differential equations [58, 66].

2.2.1 The L^p-space

If $X : \Omega \to \mathbb{R}^n$ is a random variable and $p \in [1, \infty)$ is a constant, we can define the L^p-norm of X, $\|X\|_p$, by

$$\|X\|_p = \|X\|_{L^p(\Omega,\mathcal{F},P)} = \left(\int_\Omega |X(\omega)|^p dP(\omega)\right)^{\frac{1}{p}}. \tag{2.3}$$

If $p = \infty$ we set

$$\|X\|_\infty = \|X\|_{L^\infty(\Omega,\mathcal{F},P)} = \text{ess sup}\{|X(\omega)|; \omega \in \Omega\}. \tag{2.4}$$

The corresponding L^p- spaces are defined by

$$L^p(\Omega, \mathcal{F}, P) = \{X : \Omega \to \mathbb{R}^n; \|X\|_p < \infty\}. \tag{2.5}$$

With the above norm the space $L^p(\Omega, \mathcal{F}, P)$ is a Banach space, i.e. a complete normed linear space. If $p = 2$ the space $L^2(\Omega, \mathcal{F}, P)$ is even a Hilbert space, i.e. a complete inner product space, with inner product

$$(X_1, X_2)_{L^2(\Omega,\mathcal{F},P)} := E(X_1 \cdot X_2); \quad X_1, X_2 \in L^2(\Omega, \mathcal{F}, P). \tag{2.6}$$

2.2.2 Gaussian Hilbert Spaces

For centered variables $X_1, X_2 \in L^2$, the expression $\langle X_1, X_2\rangle_{L^2}$:=cov(X_1, X_2) defines a scalar product with norm $\|X\|_2^2 := \text{Var}_X$. A Gaussian Hilbert space G [66] is a subspace of $L^2(\Omega, \mathcal{F}, P)$ which consists of centered Gaussian random.

2.3 Random Fields

The stochastic process and random fields model the uncertainties in engineering or in physical quantities varying. The stochastic process expresses uncertainty in time, whereas the random field expresses uncertainties on a domain in higher dimensions. In this dissertation, we use the term random fields. For mathematical introduction to stochastic processes see [6, 35, 54, 57]. For a mathematical work on random fields see [20, 65]. For detail about the application of this area in earth sciences see [24]. Intuitively a random field is a stochastic process parametrized by one or more parameters and taking value in Euclidean space.

2.3.1 Definition of Random Fields

A random field interpret as a set of random variables or as a function-valued random variable which is denoted by X. In both situations, a random field is a measurable mapping

$$X : D \times \Omega \to \mathbb{R}. \tag{2.7}$$

The random variable corresponds to a probability space see [6, 24, 65].

2.3.2 Determining Random Fields

The random field can be described as finite dimensional distribution or via a measure on a probability space. However, such description is in fact not practical in engineering problems. In practice, two models namely Gaussian and non-Gaussian are used [44, 45].

Out of these, the Gaussian random field is the most frequently used model. Gaussian model is easy to work with as linear combination of Gaussian random variable is again Gaussian and these random variable are independent. These fields can be described by their mean and covariance functions,

$$\mu(x) = E[X]. \tag{2.8}$$

$$K(s,t) = E[(X(s) - \mu(s))(X(t) - \mu(t))]. \tag{2.9}$$

Due to various applications, the modelling of non-Gaussian random fields is still an active area of research see [45, 61, 67, 69].

2.3.3 Representation as Expansions

In some studies, Gaussian or non-Gaussian random field is written as a finite sum of centered Gaussian or non-Gaussian random variables as

$$X(x,w) = E[X(x)] + \sum_{i=1}^{M} k_i(x)\psi_i(\omega), \tag{2.10}$$

where $\psi_i(\omega)$ is random variable and $k_i(\cdot) : D \to \mathbb{R}$ is deterministic function. More detailed a bout the model can be found in [13, 29, 30, 69].

Similarly, the non independent random variables can be expressed as [4]

$$X(x,w) = E[X(x)] + \sum_{i=1}^{M} k_i(x)\hat{\psi}_i(x_i,\omega),$$

where $k_i(x)$ are weighting functions and $\hat{\psi}_i(x_i,\omega)$ values at some position x_i.

2.4 Karhunen Loeve Expansion

To solve the problem numerically, the random field is required to be expressed in terms of a finite number of variables . But in case of stochastic partial differential equations the random fields can not be written in terms of a finite number of random variables. In order to solve this problem, we decompose the random field as infinite sum including eigenvalues and eigenfunctions of some linear operator. To do so, we require this operator to be compact, self adjoint and positive. Such expansion is known as Karhunen Loeve (KL) Expansion. Some of the functional analysis background [53] needed for this purpose is described in this section.

2.4.1 Compact, Self Adjoint Operators

In this section we describe how to obtain spectral decomposition of a linear operator. We require our operator to be linear, self adjoint and positive. In addition, it should be compact. For this we make the following definition

Definition 2.1 *[16] A subset S of a complete metric space is precompact if its closure is compact.*

The following proposition give two important characterizations of precompact sets.

Proposition 2.1 *[16] Suppose S is a subset of a complete metric space.*

1. S is precompact if and only if every sequence in S has a Cauchy subsequence.

2. S is precompact if and only if S can be covered by a finite number of balls of a fixed, arbitrary radius.

Using the notation of precompactness, we define the compact operator in the following definition

Definition 2.2 *[16] A linear operator $T : X \to Y$ is compact if the image TB of a unit ball B in X is precompact in Y.*

In our study, T is an integral operator. We note that if we consider X and Y are two Banach spaces of functions that map D_1 and D_2 to $\mathbb{R}$ respectively, then we can define $T : X \to Y$ as

$$Tu(\cdot) = \int_{D_1} K(x, \cdot)u(x)dx, \tag{2.11}$$

where, K is a kernel maps from $D_1 \times D_2$ to $\mathbb{R}$. The following lemma gives compactness result of Banach spaces X and Y.

Lemma 2.1 *If the kernel of (2.11) is $L^2(D_1 \times D_2)$, then* (2.11) *is compact operator from $X = L^2(D_1)$ to $Y = L^2(D_1)$.*

As we are interested in positive self adjoint operator, we give the following definitions .

Definition 2.3 *[16] Suppose T is an operator mapping a Hilbert space H with inner product $(\ ,\ )$ into itself. Then T is self adjoint if for all $u, v \in H$,*

$$(Tu, v) = (u, Tv).$$

Definition 2.4 *[16] Suppose T is a self adjoint operator on the Hilbert space H. Then T is positive if the quadratic form $(Tu, u) \geqslant 0$ for all $u \in H$.*

Based on these definitions, one can investigate the spectrum and eigenspaces of such operators. The following lemmas show that the relationship between a self adjoint operator T on the Hilbert space H and the eigenpairs of T

Lemma 2.2 *[16] Suppose T is a self adjoint operator on the Hilbert space H. Then there is an orthonormal basis ϕ_n for H consisting of eigenfunctions of T. Furthermore, the corresponding eigenvalues are real and their only point of accumulation is 0.*

Again, like spectral theory of matrices, we can state another similar result for positive operators.

Lemma 2.3 *[16] Suppose the self adjoint operator T is positive, then the eigenvalues of T lie on the nonnegative real line.*

We shall use the above results to get an expansion (KL expansion) corresponding to an appropriate operator in our problem.

2.4.2 The KL Expansion

Suppose $X \in L^2(D \times \Omega)$ is a random field. We can define the covariance function of X as

$$K(x, x') = E\left[(X(x, \cdot) - E[X(x, \cdot)])(X(x', \cdot) - E[X(x', \cdot)])\right]. \tag{2.12}$$

The linear operator T can be defined with the kernel $K(x, x')$ as

$T : L^2(D) \to L^2(D)$ such that for all $u \in L^2(D)$,

$$Tu(x') = \int_D K(x, x')u(x)dx. \tag{2.13}$$

The following theorem classifies this operator.

Theorem 2.1 *[16] The linear operator T defined by (2.13) is compact, self adjoint, and positive. Therefore, the eigenfunctions of T form an orthonormal basis of $L^2(D)$ and the eigenvalues are all nonnegative real numbers with only one accumulation point, namely* 0.

Proof: By using the Cauchy-Schwarz and the Jensens inequalities, we obtain

$$\begin{aligned}
\int_{D\times D} K(x, x')dxdx' &= \int_{D\times D} E\left[(X(x,\cdot) - E[X(x,\cdot)])(X(x',\cdot) - E[X(x',\cdot)])\right] dxdx' \\
&\leq \int_{D\times D} E\left[(X(x,\cdot) - E[X(x,\cdot)])^2\right]^{\frac{1}{2}} \\
&\times E\left[(X(x',\cdot) - E[X(x',\cdot)])^2\right]^{\frac{1}{2}} dxdx' \\
&= \left(\int_D E\left[(X(x,\cdot) - E[X(x,\cdot)])^2\right]^{\frac{1}{2}} dx\right) \\
&\times \left(\int_D E\left[(X(x,\cdot) - E[X(x,\cdot)])^2\right]^{\frac{1}{2}} dx'\right) \\
&= \left(\int_D E\left[(X(x,\cdot) - E[X(x,\cdot)])^2\right]^{\frac{1}{2}} dx\right)^2 \\
&= \|X(x,\cdot) - E[X(x,\cdot)]\|^2_{L^2(D\times\Omega)} \\
&\leq \left(\|X\|_{L^2(D\times\Omega)} + \|E[X(x,\cdot)]\|_{L^2(D\times\Omega)}\right)^2 \\
&\leq 4\|X\|^2_{L^2(D\times\Omega)}.
\end{aligned}$$

Since the random field $X \in L^2(D \times \Omega)$, the integral operator $K \in L^2(D \times D)$. Therefore, we conclude from lemma (2.1) that X is a compact operator from $L^2(D)$ to $L^2(D)$ and for all $u, v \in L^2(D)$ we get the following

$$\begin{aligned}
\int_D (Tu(x'))v(x')dx' &= \int_D \left(\int_D K(x,x')u(x)dx \right) v(x')dx' \\
&= \int_D \left(\int_D K(x,x')u(x)dx \right) v(x')dx' \\
&= \int_D u(x) \left(\int_D K(x',x)v(x')dx' \right) dx \\
&= \int_D u(x)(Tv(x))dx.
\end{aligned}$$

Hence T is a self adjoint operator.

Finally, using Fubini's theorem we obtain for all $u \in L^2(D)$ the following

$$\begin{aligned}
\int_D (Tu(x'))u(x')dx' = &\int_D \left(\int_D E[(X(x,\cdot) - E[X(x,\cdot)])(X(x',\cdot) - E[X(x',\cdot)])]u(x)dx \right) \\
&u(x')dx' \\
&= E\left[\int_D \int_D (X(x,\cdot) - E[X(x,\cdot)])(X(x',\cdot) - E[X(x',\cdot)])u(x)u(x')dxdx' \right] \\
&= E[\left(\int_D (X(x,\cdot) - E[X(x,\cdot)])u(x)dx \right) \\
&\times \left(\int_D (X(x',\cdot) - E[X(x',\cdot)])u(x')dx' \right)] \\
&= E\left[\left(\int_D (X(x,\cdot) - E[X(x,\cdot)])u(x)dx \right)^2 \right] \\
&\geq 0.
\end{aligned}$$

Hence, T is positive.

We thus conclude that the eigenvalues of T are real, positive numbers with accumulation point 0 and the corresponding eigenfunctions form an orthonormal basis of $L^2(D)$.$\square$

We can assume without lose of generality that $E[X(x,\cdot)] = 0$. If $E[X] \neq 0$, then we consider the random field $X(x,\cdot) - E[X(x,\cdot)]$ and use the corresponding kernel and integral operator. Now, let (λ_n, a_n) denote the eigenpair of the operator T. Since $\{a_n\}$ is an orthonormal basis of $L^2(D)$ the random field X has the following spectral decomposition

$$X(x,\omega) = \sum_{n=1}^{\infty} a_n(x)\psi_n(\omega). \tag{2.14}$$

We multiply both sides by $a_m(x)$, integrate over D and use Lebesgue Dominated convergence theorem as follows

$$\begin{aligned}
\int_D a_m(x)X(x,\omega) &= \int_D a_m(x)\sum_{n=1}^{\infty} a_n(x)\psi_n(\omega)dx \\
&= \int_D \sum_{n=1}^{\infty} \psi_n(\omega)a_m(x)a_n(x)dx \\
&= \sum_{n=1}^{\infty} \int_D \psi_n(\omega)a_m(x)a_n(x)dx \\
&= \psi_m(\omega).
\end{aligned}$$

The above formula gives the coefficient $\psi_m(\omega)$ in the expansion (2.14). As these coefficients are orthogonal, and $\{a_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(D)$, we get for $m \neq n$

$$
\begin{aligned}
E[\psi_m \psi_n] &= E\left[(\int_D a_m(x) X(x, \cdot) dx)(\int_D a_n(x') X(x', \cdot) dx') \right] \\
&= \int_D \left(\int_D E[X(x, \cdot) X(x', \cdot)] a_m(x) dx \right) a_n(x') dx' \\
&= \int_D \lambda_m a_m(x) a_n(x') dx' \\
&= 0,
\end{aligned}
$$

Also, the expected value of these coefficient equal to zero.

$$
\begin{aligned}
E[\psi_m] &= E\left[\int_D X(x, \cdot) a_m(x) dx \right] \\
&= \int_D E[X(x, \cdot)] a_m(x) dx \\
&= 0,
\end{aligned}
$$

since $E[X(x, \cdot)] = 0$. Now, to normalize the random coefficients ψ_n, we consider

$$
\begin{aligned}
E[\psi_m^2] &= E\left[\left(\int_D X(x, \cdot) a_m(x) dx \right) \left(\int_D X(x', \cdot) a_m(x') dx' \right) \right] \\
&= E\left[\int_D \int_D X(x, \cdot) a_m(x) dx X(x', \cdot) a_m(x') dx' \right] \\
&= \int_D \int_D E[X(x, \cdot) X(x', \cdot)] a_m(x) dx a_m(x') dx'] \\
&= \int_D (Tu(x')) a_m(x') dx' \\
&= \int_D \lambda_m a_m^2(x') dx' \\
&= \lambda_m .
\end{aligned}
$$

Thus, the random variables $\xi_m = \frac{1}{\sqrt{\lambda_m}} a_m$ from an orthonormal set in $L^2(\Omega)$ with zero mean. The expansion of X is defined by

$$X(x, \omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} a_n(x) \xi_n(\omega).$$

In case, $E[X(x, \cdot)]$ is not zero, we can still compute a_n and λ_n using $\hat{X}(x, \omega) = X(x, \omega) - E[X(x, \cdot)]$ to obtain the following expansion

$$X(x, \omega) - E[X(x, \cdot)] = \sum_{n=1}^{\infty} \sqrt{\lambda_n} a_n(x) \xi_n(\omega).$$

In order to define ξ_n, notice that

$$\begin{aligned} \xi_n(\omega) &= \frac{1}{\sqrt{\lambda_n}} \int_D a_n(x) \hat{X}(x, \omega) dx, \\ &= \frac{1}{\sqrt{\lambda_n}} \int_D a_n(x)(X(x, \omega) - E[X(x, \cdot)]) dx. \end{aligned}$$

From the above results the following theorem construct the formula of the KL expansion

Theorem 2.2 *[16] For a random field $X \in L^2(D \times \Omega)$, the Karhunen-Loeve expansion of X is given by:*

$$X(x, \omega) = E[X(x, \cdot)] + \sum_{n=1}^{\infty} \sqrt{\lambda_n} a_n(x) \xi_n(\omega), \tag{2.15}$$

where a_n and λ_m are the eigenfunctions and eigenvalues of the compact, self adjoint, positive operator 2.12) where $\{\xi_n\}$ are an orthonormal basis of $L^2(\Omega)$, $\{\lambda_n\}$

are positive and real and $\{\xi_n\}$ *are given by*

$$\xi_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_D a_n(x)(X(x,\omega) - E[X(x,\cdot)])dx.$$

Furthermore, ξ_n satisfies the following:

1. $E[\xi_n] = 0$ for all n.

2. $E[\xi_n \xi_m] = 0$ for all $n \neq m$.

3. $E[\xi_n^2] = 1$ for all n.

Thus, $\{\xi_n\}$ are uncorrelated random variables which has mean equal zero and variance equal one.

2.5 Stochastic Galerkin Method

In this section, our aim is to introduce the main ideas behind the stochastic finite element method, sFEM, and apply it to the model problem presented in the previous chapter. The (deterministic) finite element method, FEM, can be characterized to be a method for converting a continuum valued problem, such as a partial differential equation, into a discrete problem. The differential equation is first presented in a variational form, that is, the equation is required to hold only in a weak sense; an equation holds in a weak sense if it holds with respect to suitable test vectors or test functions. Formulating the problem in the variational form is in essence the same as to require a solution to the original problem in the sense of distributions.

The stochastic Galerkin methods are methods for which discretization with respect to space parameter is also effected using a Galerkin approach. The Galerkin finite element method (GFEM) is a widely used FEM whose key property is that the error of the weak solution is orthogonal to the corresponding FEM solution space in the sense of energy inner product. For more information about the deterministic FEM, we refer to [11]. The idea of the sFEM is the same as that of the FEM: formulate the problem in a variational sense, use a finite set of basis functions to discretize the problem, and finally solve the discretized problem to obtain an approximate solution to the original problem. However, in sFEM the problem is discretized in both, random and spatial, dimensions, or more precisely, in both parameter spaces: the discretization in the spatial dimension is done in the same way as in the deterministic FEM, and the discretization in the random dimension is usually performed using the expansion introduced in Chapter 4.

The idea in the sGFEM is to build on top of the GFEM by adding the expansion discretization of the random dimension over the standard discretization of the spatial dimension. For more information about the sFEM , see [40, 62, 63].

CHAPTER 3

AN ORTHONORMAL BASIS OF MARTINGALE SUBSPACES

We begin this Chapter by introducing an orthonormal basis for certain Martingale subspaces. Section 3.1 discusses a basis of exponential martingales. This is followed by, introduction of the basis that is used throughout the thesis. In this basis we can represent the random field $A^{-1}(x, \omega)$ in the problem (1.1) with appropriate representation. In particular, we define the basis in $L^2(\Omega, \mathcal{F}, P)$, for a special σ-algebra $\mathcal{F}$ in Section 3.2. Finally, in Section 3.3 we discuss the connection with Darcy's equation.

3.1 Bases of Exponential Martingales

Let $(\Omega, \mathcal{F}_T, P)$ be a measure space where $\mathcal{F}_T$ is the σ-algebra generated by the n-dimensional Brownian motion $\{B_s\}_{0 \leq s \leq T}$. The exponential martingales are defined by the following lemma [6],

Lemma 3.1 *The linear span of random variables of the type*

$$F_t = \exp\left\{\int_0^t f_s \cdot dB_t - \frac{1}{2}\int_0^t f_s^2 ds\right\} \quad 0 \le t \le T, \tag{3.1}$$

is dense in $L^2(\Omega, \mathcal{F}_T, P)$. *where* $f(t) = (f_1(t), f_2(t), ..., f_n(t))$, $f_n \in L^2[0,T]$, $f^2 = f \cdot f$.

Remarks:

1. F_t is called an exponential martingale , and it has the property:

$$dF_t = F_t f(t) dB_t.$$

2. To avoid technical details we will take $T = 1$ and $\mathcal{F}_T = \mathcal{F}_1 = \mathcal{F}$.

Some relevant properties of the martingales F_t are stated in the following lemmas.

Lemma 3.2

$$E[F_t^\alpha] = \exp\left(\frac{1}{2}(\alpha^2 - \alpha)\int_0^t f_s^2 ds\right),$$

where, F_t^α *is an exponential martingale and* α *is a constant.*

Proof. Let $V_t = F_t^\alpha$. Then, by Ito formula,

$$\begin{aligned} dV_t &= \alpha F_t^\alpha f_t dB_t - \frac{1}{2}\alpha F_t^\alpha f_t^2 dt + \frac{1}{2}\alpha^2 F_t^\alpha f_t^2 dt \\ &= \alpha V_t f_t dB_t + \frac{1}{2}(\alpha^2 - \alpha) V_t f_t^2 dt, \end{aligned}$$

so that

$$E[V_t] = 1 + \frac{1}{2}(\alpha^2 - \alpha)\int_0^t E[V_s] f_s^2 ds.$$

Putting $y(t) = E[V_t]$, we have the deterministic differential equation

$$\frac{dy}{dt} = \frac{1}{2}(\alpha^2 - \alpha) y f_t^2, \quad y(0) = 1,$$

whose solution is

$$y = \exp\left(\frac{1}{2}(\alpha^2 - \alpha)\int_0^t f_s^2 ds\right). \square$$

Corollary 3.1

$$E[F_t] = 1,\ E[F_t^2] = \exp\left(\int_0^t f_s^2 ds\right).$$

Proof. As a result of Lemma 3.2 when $\alpha = 1$ and $\alpha = 2$ we show that.

Lemma 3.3

$$E[F_t G_t] = \exp\left(\int_0^t f_s \cdot g_s ds\right).$$

Proof. Let $V_t = F_t G_t$. Then, by Ito formula,

$$\begin{aligned} dV_t &= F_t dG_t + G_t dF_t + dF_t dG_t \\ &= F_t G_t g_t dB_t + F_t G_t f_t dB_t + F_t G_t f_t dB_t g_t dB_t \\ &= V_t(g_t + f_t) dB_t + V_t f_t dB_t dB_t^T g_t^T \\ &= V_t(f_t + g_t) dB_t + V_t f_t \cdot g_t dt, \end{aligned}$$

so that

$$E[V_t] = 1 + \int_0^t E[V_s] f_s \cdot g_s ds.$$

Putting $y(t) = E[V_t]$, we have the deterministic differential equation

$$\frac{dy}{dt} = y f_t \cdot g_t, \quad y(0) = 1,$$

whose solution is

$$y = e^{\int_0^t f_s \cdot g_s ds} \square$$

This yields the following orthogonality result.

Corollary 3.2 *If f, g are mutually orthogonal in $(L^2[0,1])^n$, then $(F_t - 1), (G_t - 1)$ are mutually orthogonal in $L^2(\Omega, \mathcal{F}, P)$ (i.e, F_t, G_t are uncorrelated).*

Proof. Assuming f, g are mutually orthogonal,

$$\begin{aligned} E[(F_t - 1)(G_t - 1)] &= E[F_t G_t] - E[F_t] - E[G_t] + 1 \\ &= e^{\int_0^t f_s \cdot g_s ds} - 1 - 1 + 1 \\ &= 1 - 1 - 1 + 1 = 0 \square \end{aligned}$$

Lemma 3.4

$$E[F_t G_t H_t] = \exp\left(\int_0^t f \cdot g + g \cdot h + h \cdot f ds \right).$$

Proof. Let $Z_t = F_t G_t H_t$, then by Ito formula

$$\begin{aligned} dZ_t &= F_t G_t dH_t + F_t H_t dG_t + G_t H_t dF_t + F_t dG_t dH_t + H_t dG_t dF_t + G_t dH_t dF_t \\ &= Z_t[h dB_t + g dB_t + f dB_t + g \cdot h dt + f \cdot g dt + h \cdot f dt] \\ &= Z_t(f+g+h) dB_t + Z_t(f \cdot g + g \cdot h + h \cdot f) dt, \end{aligned}$$

so that

$$E[Z_t] = 1 + \int_0^t E[Z_s](f \cdot g + g \cdot h + f \cdot h) ds.$$

Putting $y(t) = E[Z_t]$, we have the deterministic differential equation

$$\frac{dy}{dt} = y(f \cdot g + g \cdot h + h \cdot f), \quad y(0) = 1,$$

whose solution is

$$y = \exp\left(\int_0^t (f \cdot g + g \cdot h + f \cdot h) ds\right) \square$$

Corollary 3.3 *If f, g, h are pairwise orthogonal in $(L^2[0,1])^n$, then*

$E[(F_t - 1)(G_t - 1)(H_t - 1)] = 0.$

Proof. Suppose f, g, h are mutually orthogonal in $(L^2[0,1])^n$. then, taking into account the orthogonality of f, g, h,

$$
\begin{aligned}
E[(F_t-1)(G_t-1)(H_t-1)] &= E[F_tG_tH_t] - E[F_tG_t] - E[G_tH_t] - E[F_tH_t] + E[F_t] \\
&+ E[G_t] + E[H_t] - 1 \\
&= \exp\left(\int_0^1 (f\cdot g + g\cdot h + h\cdot f)ds\right) - \exp\left(\int_0^1 (f\cdot g)ds\right) \\
&- \exp\left(\int_0^1 (g\cdot h)ds\right) - \exp\left(\int_0^1 (h\cdot f)ds\right) + 1 + 1 + 1 - 1 \\
&= 1 - 1 - 1 - 1 + 1 + 1 + 1 - 1 = 0. \square
\end{aligned}
$$

Lemma 3.5 *Let $f_k \in (L^2[0,1])^n$ for $k = 1, ..., n$, then f_k convergence to f in $(L^2[0,1])^n$ if and only if F_k convergence to F in $L^2(\Omega, \mathcal{F}, P)$.*

Proof. Suppose $f_k \to f$. Then

$$
\begin{aligned}
E[(F_k - F)^2] &= E[F_k^2] - 2E[F_kF] + E[F^2] \\
&= e^{\int_0^1 f_k^2 ds} - 2e^{\int_0^1 f\cdot f_k ds} + e^{\int_0^1 f^2 ds} \to 0, \ as \ k \ \to \infty.
\end{aligned}
$$

Similarly, suppose $F_k \to F$. Then

$\int_0^1 (f_k - f)^2 ds = \log E[F_k^2] - 2\log E[F_k^2 F] + \log E[F^2] \to 0$, as $k \to \infty$. $\square$

Now let $\{f_k\}_{k=1}^{\infty}$ be a dense sequence of functions in $(L^2[0,1])^n$ and let $G_k = F_k - 1$. By Lemma 3.4 and corollary 3.3, the linear span of the functions $\{1, G_1, G_2, ...\}$ is dense in $L^2(\Omega, \mathcal{F}, P)$. Furthermore, if the functions f_k are orthogonal, then the random variables G_k are orthogonal.

Theorem 3.4 *The linear span of random variables of the type*

$$Z_h = \exp\left\{\int_0^1 h(t)dB_t(w) - \frac{1}{2}\int_0^1 h^2(t)dt\right\} - 1, \tag{3.2}$$

is dense in $L^2(\Omega, \mathcal{F}, P)$*. where,* $h \in (L^2[0,1])^n$ *is deterministic.*

The proof of this theorem can be found in [6].

Theorem 3.5 *Let* φ *be a dense subset of functions in* $(L^2[0,1])^n$*. Then* $\{Z_h : h \in \varphi\}$ *is dense in* $L^2(\Omega, \mathcal{F}, P)$*.*

Proof. Let $f \in L^2(\Omega, \mathcal{F}, P)$, $\epsilon > 0$. By Theorem 3.4, there exist $\{h_1, h_2, ..., h_l\}$ such that $\|\sum_{i=1}^l \alpha_i Z_{h_i} - f\| < \epsilon$. For each i there exist a sequence $q_k^{(i)} \in \varphi$ such that $q_k^{(i)} \to h_i$. Then

$Z_{q_k}(i) \to Z_{h_i}$ as $k \to \infty$ implying that $\sum_{i=1}^l \alpha_i Z_{q_k}(i) \to \sum_{i=1}^l \alpha_i Z_{h_i}$.□

3.2 A Hierarchal Basis for $L^2[0,1]$

Let Φ_j be a basis for a subspace $V_j \in H$, where H is a Hilbert space. The sequence $\{\Phi_j\}_{j \in J}$ is called a Hierarchal basis if

(i) $\Phi_j \subset \Phi_{j+1}$,

(ii) $\overline{\cup_{j\in J} \operatorname{span} \Phi_j} = H$. In this section we construct a hierarchal basis $H_{j,k}$ for $L^2[0,1]$. This will enable us to write $L^2[0,1] = \bigcup_{k=1}^{\infty} V_j$ where $V_j = \operatorname{span}\{H_{j,k}\}_{k \in I_j}$ and $I_j =, 1, 2, ..., 2^j$.

Definition 3.1 *A dyadic interval is an interval of the type*

$$I_{j,k} = \left[2^{-j}k, 2^{-j}(k+1)\right).$$

A dyadic step function with scale j is a function which is constant on each interval $I_{j,k}$(with j fixed). For a fixed j, $I_{j,k} \subset [0,1]$, $k \in I_j$.

Definition 3.2 *The haar scaling functions of order j are given by:*

$$H_{j,k} = 2^{j/2} H(2^j x - k), \;\; k \in I_j.$$

where

$$H(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

Lemma 3.6 *We have the alternative formula*

$$H_{j,k}(x) = \begin{cases} 1, & x \in I_{j,k} \\ 0, & otherwise \end{cases}.$$

Proof. $H_{j,k}(x) \neq 0 \iff 0 \leq 2^j x - k < 1 \iff k \leq 2^j x < k+1 \iff x \in [2^{-j}k, 2^{-j}(k+1)) = I_{j,k}.\square$

Notation:

$V_j = \{$set of all dyadic step functions of scale j with support in $L^2[0,1]\}$. Then, for any $f \in V_j$,

$$f(x) = \sum_{k \in I_j} \alpha_k H_{j,k}(x).$$

Definition 3.3 *We define the projection* $H_j : L^2[0,1] \longrightarrow V_j$ *by*

$$H_j f = \sum_{k \in Ij} (f, H_{j,k}) H_{j,k}.$$

Theorem 3.6 *For any* $j \in \mathbb{Z}$*, the set of functions* $\{H_{j,k}\}_{k=-\infty}^{\infty}$ *is an orthonormal basis for* V_j.

Proof. Fix $j \in \mathbb{Z}$. Consider $(j,k), (j,k')$ be two pairs of indices. If $k \neq k'$, then $I_{j,k} = [2^{-j}k, 2^{-j}(k+1))$, $I_{j,k'} = [2^{-j}k', 2^{-j}(k'+1))$. If $k' > k+1$, then,

$$2^{-j}k' > 2^{-j}(k+1) \,\text{and}\, I_{j,k} \cap I_{j,k'} = \phi, \text{thus}\, (H_{j,k}, H_{j,k'}) = 0.$$

If $k = k'$ then

$$(H_{j,k}, H_{j,k}) = \|H_{j,k}\|^2 = 2^j \int_{I_{j,k}} dx = 1.$$

Therefore, $\{H_{j,k}\}_{-\infty}^{\infty}$ is an orthonormal basis for V_j. □

Lemma 3.7 $\{H_{j,k}\}_{j=-\infty}^{\infty}, k \in I_j$ *is a Hierarchal basis for* $L^2[0,1]$.

As a corollary of theorem 3.4 we get,

Corollary 3.7 *The Linear span of random variables of the type*

$$Z_{j,k} = \exp\left\{2^{j/2}(B_{2^{-j}(k+1)} - B_{2^{-j}k}) - 1/2\right\} - 1, \tag{3.3}$$

is hierarchal basis in $L^2(\Omega, \mathcal{F}, P)$.

3.3 Application to Darcy Equation

For the Darcy equation

$$\nabla \cdot (A\nabla u) = 0, \tag{3.4}$$

in two (or three) dimensions with appropriate assumptions, if we assume that A is $\mathcal{F}$ measurable, then u will also be $\mathcal{F}$ measurable. Thus, the basis constructed in the last section can be used to expand both u and A: Intuitively, A is thought of as an evolved Brownian motion where its covariance at each time instant and each $x \in D$ is determined by a function $\sigma(t, x, A)$ and its mean by $b(t, x, A)$ where $b : [0, T] \times D \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, T] \times D \times \mathbb{R} \to \mathbb{R}^{1+n}$. This means that A satisfies the SDE

$$dA = b(t, x, A)dt + \sigma(t, x, A)dB_t, \tag{3.5}$$

where B_t is an n-dimensional Brownian motion. If the initial conditions are deterministic or $\mathcal{F}$ adapted, then A is $\mathcal{F}$ measurable.

The model

$$dA = Ab(t, x)dt + A\sigma(t, x)dB_t,$$

has the solution

$$A_t = A_0 \mathrm{exp} \left(\int_0^t \sigma dB_s + \int_0^t \{b - \frac{1}{2}\sigma\sigma^T\} \right),$$

which satisfies the positivity condition.

CHAPTER 4

CONVERGENCE ANALYSIS OF STOCHASTIC GALERKIN MIXED APPROXIMATIONS OF DARCY'S PROBLEM

In this Chapter we study the Darcy equation and we also study the stochastic Galerkin mixed approximation and its convergence analysis. First, we give a brief overview of the model. In particular, in Section 4.1 we focus on Darcy's equation and its mixed variational formulation with data uncertainty. In Section 4.2 we describe the stochastic saddle point problem corresponding to the problem. A perturbed stochastic saddle point problem will also be discussed in Section 4.3. The Galerkin approximation will be introduced in Section 4.4. Finally, in Section 4.5 we will investigate the error analysis.

4.1 An Overview.

The numerical approximation of solutions to stochastic partial differential equations is an active research area [3, 28, 31, 32, 33, 46, 55]. Due to its application, the research in stochastic finite element methods (SFEMS) for solving partial differential has attracted many authors as in above references. However, the stochastic saddle point problem has not attracted much attention [5, 8, 26]. In this Chapter we focus on stochastic saddle point method, present its analysis and Galekin approximation. The error analysis in perturbation as well as stochastic Galerkin approximation is also discussed.

Now, we consider the weak formulation: find $(u, p) \in V \times W$ such that

$$\begin{aligned} a(u,r) + b(r,p) &= l(r) \qquad r \in V, \\ b(u,v) &= h(v) \qquad v \in W, \end{aligned} \tag{4.1}$$

where the bilinear forms $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $b(\cdot,\cdot) : V \times W \to \mathbb{R}$ are bounded. Also, the linear functionals $l : V \to \mathbb{R}$, $h : W \to \mathbb{R}$ are bounded , and V and W are Hilbert spaces. The results of the general theory of saddle point problems [1] ensure a uniqueness of the solution if the above bilinear $a(\cdot,\cdot)$ is coercive on the null-space of the bilinear $b(\cdot,\cdot)$, and if the bilinear $b(\cdot,\cdot)$ satisfy the inf-sup condition

$$\exists \ \beta > 0 \quad \text{such that} \quad \sup_{r \in V} \frac{b(r,v)}{\|r\|_V} \geq \beta \|v\|_W \quad \forall v \in W, \tag{4.2}$$

where β is the inf sup-constant. The mixed formulation of boundary value problem of the problem (4.1)

$$
\begin{aligned}
A^{-1}\vec{u} + \nabla p = 0, &\qquad \text{in } D, \\
\nabla \cdot \vec{u} = -f &\qquad \text{in } D, \\
p = g &\qquad \text{on } \partial D_{Dir}, \\
\vec{n} \cdot \vec{u} = 0 &\qquad \text{on } \partial D_{Neu}.
\end{aligned}
\tag{4.3}
$$

In this case consider $V = H_0(\mathrm{div}, D)$, $W = L^2(D)$, and $a(u, r) := \int_D A^{-1}u \cdot r dx$, $b(r, p) := -\int_D p\nabla \cdot r dx$, $h(v) = \int_D f v dx$, and $l(r) = 0$ where,

$$H_0(\mathrm{div}, D) := \left\{r \in L^2(D) : \mathrm{div}\, r \in L^2(D),\ \langle r, \nabla v\rangle + \langle \nabla \cdot r, v\rangle = 0\ \forall v \in H^1_{0,Dir}(D)\right\}$$

and the space $H^1_{0,Dir}(D)$ contains $H^1(D)$ which consists of functions with vanishing trace on ∂D_{Dir}. We use the usual notation $\nabla = (\partial/\partial_{x_1}, ..., \partial/\partial_{x_d}), \mathrm{div} = \nabla\cdot$ to define the differential operators. The Laplace operator $\Delta = \nabla \cdot \nabla = \mathrm{div}\nabla$. For $V = H_0(\mathrm{div}, D)$ and $W = L^2(D)$. The inf-sup condition (4.2) is satisfied [1] and thus we can find inf sup constant $\beta > 0$, which depends upon the domain D, such that,

$$\sup_{r\in H_0(\mathrm{div},D)} \frac{\int_D v\, \mathrm{div}\, r dx}{\|r\|_{H_0(\mathrm{div},D)}} \geq \beta\|v\|_{L^2(D)} \quad \forall v \in L^2(D). \tag{4.4}$$

To obtain the Galerkin approximations to the weak solution (u, p) of the problem (4.1), we replace V and W with finite dimensional subspaces. To this end, we use Raviart Thomas(RT) elements with respect to D to introduce the two finite element spaces.

As it frequently happens in engineering applications, we now suppose that A^{-1} in (4.3) is not exactly known everywhere in our domain of computation (see also Chapter 6). To deal with the data with uncertainty we model A^{-1} as a random field over a complete probability space $(\Omega, \mathcal{F}, P)$. In here, Ω is the trial space, $\mathcal{F}$ is a special σ-algebra generating by random variables in A^{-1} and the probability measure is given by $P : \mathcal{F} \to [0, 1]$. Thus when $A^{-1} = A^{-1}(x, \omega)$ where $x \in D$, $\omega \in \Omega$, the solution to (4.3) is $(u, p) = (u(x, \omega), p(x, \omega))$ such that, P-a.e. in Ω,

$$\begin{aligned}
A^{-1}\vec{u}(x,\omega) + \nabla p(x,\omega) &= 0 && \text{in } D, \\
\nabla \cdot \vec{u}(x,\omega) &= -f(x,\omega) && \text{in } D, \\
p(x,\omega) &= g(x,\omega) && \text{on } \partial D_{Dir}, \\
\vec{n} \cdot \vec{u}(x,\omega) &= 0 && \text{on } \partial D_{Neu}.
\end{aligned} \tag{4.5}$$

The above variational problem (4.5) is of the same form as (4.1) expect that the bilinear form $a(\cdot, \cdot)$ contains a random coefficient and the solution and test functions are not determinate but are random fields. Approximations of (4.5) are found by stochastic Galerkin methods in [8, 26, 50].

4.2 Stochastic saddle point problem.

To construct the saddle point problem (4.5). We introduce some notations and describe the suitable spaces of random fields. $L^q_P(\Omega)$ is the set of real-valued random variables that are integrable over the probability space Ω. In order that the mean $E(\xi)$ is defined for all $\xi \in L^1_P(\Omega)$ as

$$E[\xi] = \int_\Omega \xi(\omega) dP(\omega) = \int_\mathbb{R} y \rho_\xi(y) dy,$$

where ρ_ξ is the probability density function for ξ. The definition of the covariance of two random variables can be given as

$$\mathrm{Cov}(\xi, \eta) = E\left[(\xi - E[\xi])(\eta - E[\eta])\right],$$

where, $\xi, \eta \in L^2_P(\Omega)$. In what follows we will use $E[\cdot]$ or $\langle \cdot \rangle$ to denote expectation interchangeably. For any function in D, the inner product space $L^2(D)$ has inner product $(\cdot, \cdot)$ and the corresponding norm $\| \cdot \|$. Also, the space of vector valued functions is denoted by $p \in (L^2(D))^d$ is equipped . with the norm $\| \cdot \|_X$. The definition of the space $L^2_P(\Omega, X(D))$ as

$$L^2_P(\Omega, X(D)) = \{v : D \times \Omega \to \mathbb{R} : E[\|v\|^2_X] < \infty\},$$

where

$$\|v\|_{L^2_P(\Omega, X(D))} := (E[\|v\|^2_X])^{\frac{1}{2}}.$$

V and W are two Hilbert spaces of vector-valued random fields defined as

$$V = L^2_P\left(\Omega, H_0(\mathrm{div}, D)\right) \quad \text{and} \quad W = L^2_P(\Omega, L^2(D)),$$

The norm in these spaces is defined as

$$\|r\|_V := \left(E\left[\|r\|^2_{H_0(\mathrm{div},D)}\right]\right)^{\frac{1}{2}} \quad \text{and} \quad \|v\|_W := \left(E\left[\|v\|^2\right]\right)^{\frac{1}{2}},$$

where

$$\|r\|^2_{H(\mathrm{div},D)} := \|r\|^2 + \|\mathrm{div}\, r\|^2 .$$

The following assumption is made for the boundedness of the random field $A^{-1}(x, \omega)$ in (4.5).

Assumption 4.1 *[1] $A^{-1}(x, \omega) \in L^\infty_P(D \times \Omega)$ is uniformly bounded away from zero i.e, there exist positive constants $C_{\min}$ and $C_{\max}$ such that*

$$0 < C_{\min} \leq A^{-1}(x, \omega) \leq C_{\max} < \infty \quad a.e \text{ in } D \times \Omega. \tag{4.6}$$

We note that due to Assumption 4.1 we can refer $A^{-1}(x, w) : D \times \Omega \to \mathbb{R}$ is square integrable on Ω, i.e, $A^{-1}(x, \cdot) \in L^2_P(\Omega)\ \forall x \in D$, with the mean
$E[A^{-1}](x) := \int_\Omega A^{-1}(x, \omega) dP(\omega) \in L^2(D)$. The two bilinear forms in (4.1) are defined as

$$a(u, r) := \left\langle \int_D A^{-1}(x, \cdot) u(x, \cdot) \cdot r(x, \cdot) dx \right\rangle, \qquad u, r \in V, \tag{4.7}$$
$$b(r, v) := \left\langle \int_D v(x, \cdot) \mathrm{div}\, r(x, \cdot) dx \right\rangle, \qquad r \in V, v \in W, \tag{4.8}$$

and, if $g = 0$ on the ∂D_{Dir} in (4.5), then for a given $f \in L^2_P(\Omega, L^2(D))$, the variational formulation of problem (4.5) reads: find $(u,p) \in V \times W$ such that

$$\begin{aligned} a(u,r) + b(r,p) &= 0 \qquad && \forall r \in V, \\ b(u,v) &= -\langle (f,v) \rangle && \forall v \in W. \end{aligned} \tag{4.9}$$

The following lemma establish the well-posedness of (4.9).

Lemma 4.1 . *If $A^{-1}(x,\omega)$ satisfies assumption 4.1, then problem (4.9) has a unique solution $(u,p) \in V \times W$. Also,*

$$\begin{aligned} \|u\|_{L^2_P(\Omega, L^2(D))} &\leq \frac{C_{\max}}{\beta C_{\min}} \|f\|, \\ \|u\|_V &\leq \left(\left(\frac{C_{\max}}{\beta C_{\min}} \right) + 1 \right)^{\frac{1}{2}} \|f\|, \\ \|p\|_W &\leq \frac{C_{\max}^2}{C_{\min}\beta^2} \|f\|, \end{aligned}$$

where $C_{\min}, C_{\max}$ are as in (4.6) and $\beta > 0$ satisfies, (4.4) which is the inf-sup constant for (4.9).

The proof of this lemma is based upon the theory of saddle point problems and can be found in [21].

Remark 4.1

We can establish the existence and uniqueness under a weaker assumption on $A^{-1}(x,\omega)$. In fact, we can replace (4.6) by requiring that P-a.e. in Ω holds

$$0 < C_{\min}(\omega) \leq A^{-1}(x,\omega) \leq C_{\max}(\omega) < \infty \;\; a.e \text{ in } D,$$

where, $C_{\min}$ and $C_{\max}$ are square integrable random variables.

The advantage of this assumption is that it allows for A^{-1} to be an unbounded random variable with respect to $x \in D$ see [33, 32].

4.3 A perturbed stochastic Saddle point problem.

One method is based upon transforming the stochastic saddle point problem (4.9) into one that can be solved by using the numerical methods used in the deterministic form. For this, we use an expansion of $A^{-1}(x, \omega)$ to separate x-dependence and ω-dependence, where $x \in D$ and $\omega \in \Omega$. There are a number of such representation, see [3] for a survey of these results. Herein, we focus on an expansion of the form

$$A^{-1}(x, \omega) = E[A^{-1}] + \lim_{M \to \infty} \sum_{k=0}^{M} a_k(x) Z_{M,k}(\omega), \tag{4.10}$$

where $\{Z_{M,k}\}_{M=1,\dots,k=1,\dots,M}$ is a sequence of Hierarchal basis functions in the sense that the corresponding spaces $H_M = \text{span}\{Z_{M,k}\}_{k=1}^{M}, M = 1, 2, \ldots$ from nested sequence with dense union in a suitable space of square integrable martingales (see Chapter 3 Sec.2). In our work $Z_{M,k}$ is defined as

$$Z_{M,k} = \exp\left\{2^{\frac{M}{2}}\left(B_{2^{-M}(k+1)} - B_{2^{-M}k}\right) - \frac{1}{2}\right\} - 1, \tag{4.11}$$

where $\left(B_{2^{-M}(k+1)} - B_{2^{-M}k}\right)$ is an increment Brownian motion with mean zero and $(e-1)$ variance. By truncating A^{-1} in (4.10) after 2^M term, we obtain the approximation

$$A^{-1}(x,\omega) \approx A_M^{-1}(x,\omega) = E[A^{-1}] + \sum_{k=1}^{2^M} a_k(x) Z_{M,k}(\omega). \tag{4.12}$$

In the expression for $A^{-1}(x,\omega)$ in (4.10), we use the truncated coefficient $A_M^{-1}(x,\omega)$ to obtain a perturbed problem: find $u^{(M)} \times p^{(M)} \in V \times W$ such that

$$\begin{aligned} a_M(u^{(M)}, r) + b(r, p^{(M)}) &= 0 \qquad && \forall r \in V, \\ b(u^{(M)}, v) &= -\langle (f, v) \rangle && \forall v \in W. \end{aligned} \tag{4.13}$$

The bilinear forms $a_M(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a_M(u, r) = \left\langle \int_D A_M^{-1}(x,\cdot) u(x,\cdot) \cdot r(x,\cdot) dx \right\rangle \quad \forall u, r \in V, \tag{4.14}$$

$$b(r, v) = -\left\langle \int_D v(x,\cdot) \operatorname{div} r(x,\cdot) dx \right\rangle \quad \forall r \in V,\, v \in W. \tag{4.15}$$

Assumption 4.2 *There exist a sequence $\epsilon_M \longrightarrow 0$ as $M \longrightarrow \infty$ and as $M_0 > 0$ such that for $M > M_0$.*

$$\|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)} = \| \sum_{k=2^M+1}^{\infty} a_k(x) Z_k(\omega) \| \le C\epsilon_M,$$

where

$$L_P^\infty(D\times\Omega) = \{v : D\times\Omega \to \mathbb{R};\ \operatorname{ess\,sup}|v| < \infty\}.$$

Lemma 4.2 *If $A^{-1}(x,\omega)$ satisfy Assumptions 4.1 and 4.2, then for any $M \geq M_0$,*

$$0 < \alpha_{\min} \leq A_M^{-1}(x,\omega) \leq \alpha_{\max} \quad a.e \ in \quad D \times \Omega, \tag{4.16}$$

with $\alpha_{\min} = C_{\max} - C\epsilon_M$ and $\alpha_{\max} = C_{\max} + C\epsilon_M$, where C_{min} and C_{max} are as in (4.6).

Proof. Let M_0 be as in Assumption 4.2. Then

$$\|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)} \leq C\epsilon_M,$$

implies that $\|A^{-1}\| - C\epsilon_M \leq \|A_M^{-1}\| \leq \|A^{-1}\| + C\epsilon_M$. Using (4.6) we get,

$$\alpha_{\min} := C_{\min} - C\epsilon_M < A^{-1} - \|A^{-1} - A_M^{-1}\| \leq A^{-1} + (-A^{-1} + A_M^{-1}) = A_M^{-1}.$$

Thus,

$$\alpha_{\min} < A_M^{-1} \leq \|A_M^{-1}\| \leq C_{\max} + C\epsilon_M := \alpha_{\max}.$$

Therefore,

$$0 < \alpha_{\min} \leq A_M^{-1}(x,\omega) \leq \alpha_{\max} \ \text{a.e on } D \times \Omega. \ \square \tag{4.17}$$

We will assume that M is sufficiently large to obtain good approximation of A^{-1}. Since M will then be fixed, we will use the notation Z_k instead of $Z_{M,k}$. The following Definition introduce the definition of the $H^l(D)$ space [38]; for l non integer we adhere to the recipe of Slobodeckii [60].

Definition 4.1 *Suppose first that l is integer. We define the $H^l(D)$ and $H^l(\mathrm{div}, D)$ as*

$$H^l(D) := \{r \in (L^2(D))^d : D^s r \in L^2(D) \;\; \textit{for}\; |s| \le l\}.$$

$$H^l(\mathrm{div}, D) := \{r \in H^l(D) : \mathrm{div}\; r \in L^2(D)\}.$$

Now suppose that l is not integer, therefore $l = [l] + \lambda$, $0 < \lambda < 1$. We define $H^l(D)$ and $H^l(\mathrm{div}, D)$ as

$$H^l(D) := \{r \in (L^2(D))^d : D^s r \in L^2(D) \;\; \textit{for}\, |s| \le \epsilon \,\text{and}\; \mathrm{I}_\lambda(D^s r) < \infty)\},$$

$$H^l(\mathrm{div}, D) := \{r \in H^l(D) : \mathrm{div}\; r \in L^2(D)\}.$$

where

$$\mathrm{I}_\lambda(D^s r) := \int\int_{D\times D} \frac{|r(x) - r(y)|^2}{|x-y|^{2+2\lambda}} dxdy.$$

The norm is defined as

$$\|r\|_l^2 := \|r\|_{[l]}^2 + \sum_{|s|\le[l]} \mathrm{I}_\lambda(D^s r),$$

and the norm in $H^l(\mathrm{div}, D)$ is defined as

$$\|r\|_l^2 := \|r\|_{[l]}^2 + \|\mathrm{div}\, r\|_{[l]}^2 + \sum_{|s|\le[l]} \mathrm{I}_\lambda(D^s r).$$

Lemma 4.3 *For any $w \in W$ a vector valued function $z \in V$ exists such that*

$z \in L^2_P(\Omega, H^\epsilon(D))$ for some $\epsilon > 0$, $\operatorname{div} z = w$, and the following hold

$$\|z\|_V \leq C_D \|w\|_W, \tag{4.18}$$

$$\|z\|_{L^2_P(\Omega, H^\epsilon(D))} \leq C_{\mathrm{reg}} \|w\|_W. \tag{4.19}$$

The constant $C_D > 0$ depends on D, while the constant $C_{\mathrm{reg}} > 0$ depends on D and ϵ.

Proof.

Given $v \in W$, we consider the following: find $u \in L^2_P(\Omega, H^1_{0,\mathrm{Dir}}(D))$ such that

$$\langle \nabla u, \nabla v \rangle_W = \langle w, v \rangle_W \qquad \forall\ w \in W. \tag{4.20}$$

The uniqueness of (4.20) follows immediately from the Lax-Milgram lemma, and there holds

$$\|u\|_{L^2_P(\Omega, H^1(D))} \leq C_1 \|w\|_W, \tag{4.21}$$

where the positive constant C_1 depends on D. Moreover, using the regularity theory for deterministic elliptic problems in non-smooth domains [2, 42]. We infer that

$$\|u\|_{L^2_P(\Omega, H^{1+\epsilon}(D))} \leq C_2 \|w\|_W. \tag{4.22}$$

Now, we set $z := -\nabla u$. Then $z \in L_P^2(\Omega, H^\epsilon(D))$ and using usual arguments (see, e.g., [21], p. 136) we show that $\text{div} z = w \in W$ and

$$\langle (z, \nabla v) + (\text{div} z, v) \rangle = 0 \qquad \forall v \in L_P^2(\Omega, H_{0,Dir}^1(D)).$$

Hence, $z \in L_P^2(H^\epsilon(D) \cap H_0(\text{div}, D))$ and estimates (4.21) and (4.22) gives the desired inequalities in (4.18) and (4.19):

$$\|z\|_V = (\|\nabla u\|_W^2 + \|w\|_W^2)^{\frac{1}{2}} \leq C_D \|w\|_W,$$

where $C_D = \sqrt{1 + C_1^2}$ and

$$\|z\|_{L_P^2(\Omega, H^\epsilon(D))} \leq C_3 \|u\|_{L_P^2(\Omega, H^{\epsilon+1}(D))} \leq C_{\text{reg}} \|w\|_W, \qquad C_{\text{reg}} := C_2 C_3.$$

This complete the proof of the lemma for the function $w \in W$. $\square$

Lemma 4.4 *If A^{-1} satisfies the assumptions in Lemma 4.2 and assumption 4.2, then Problem* (4.13) *has a unique solution* $(u^{(M)}, p^{(M)}) \in \mathrm{V} \times \mathrm{W}$. *Also,*

$$\begin{aligned} \|u^{(M)}\|_V &\leq \left(\left(\frac{\alpha_{\max}}{\beta \alpha_{\min}} \right)^2 + 1 \right)^{\frac{1}{2}} \|f\|, \\ \|p^{(M)}\|_W &\leq \frac{\alpha_{\max}^2}{\alpha_{\min} \beta^2} \|f\|, \end{aligned} \tag{4.23}$$

where $\beta > 0$ is the inf-sup constant for (4.13) *in analogy with* (4.14).

proof. Following the general results of saddle point problems [1], we verify the adequate conditions for existence and uniqueness of the solution to (4.13). First, we prove continuity of the bilinear forms (4.14) – (4.15).

$$a_M(u,r) = \left\langle \int_D A_M^{-1}(x,\cdot)u(x,\cdot)\cdot r(x,\cdot)dx \right\rangle$$

By using (4.17) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |a_M(u,r)| &\leq \alpha_{\max} \left\langle \left(\int_D |u|^2 dx\right)^{\frac{1}{2}} \left(\int_D |r|^2 dx\right)^{\frac{1}{2}} \right\rangle, \\ &\leq \alpha_{\max} \|u\|_V \|r\|_V. \end{aligned} \tag{4.24}$$

Similarly, for the bilinear $b(r,v)$ we obtain that,

$$|b(r,v)| \leq \|r\|_V \|v\|_W. \tag{4.25}$$

The null space of this bilinear form $b(\cdot,\cdot)$ is given by

$$V^0 := \left\{ r \in V : b(r,v) = -\left\langle \int_D v(x,\cdot)\operatorname{div} r(x,\cdot)dx \right\rangle = 0 \quad \forall v \in W \right\}.$$

Recalling that $\operatorname{div} r \in W$ for $r \in V$, we establish the coercivity of $a_M(\cdot, \cdot)$ on V^0:

$$
\begin{aligned}
a_M(r,r) &= \left\langle \left(\int_D A^{-1}(x,\cdot)\, r(x,\cdot)\, r(x,\cdot) dx \right) \right\rangle, \\
&\geq \alpha_{\min} \left\langle \int_D r^2(x,\cdot) dx \right\rangle, \\
&= \alpha_{\min} \left\langle \|r(x,\cdot)\|_V^2 \right\rangle.
\end{aligned}
$$

Thus, we obtain that,

$$
a_M(r,r) \geq \alpha_{\min} \|r\|_V^2 \qquad \forall r \in V^0. \tag{4.26}
$$

To show that the inf-sup condition holds for $v \in W$, we use Lemma 4.3 to find $z \in V$ such that $\operatorname{div} z = v$ in W and $\|z\|_V \leq C_D \|v\|_W$, where C_D is a constant depends on D. Then, the inf-sup condition defined as

$$
\sup_{r \in V} \frac{b(r,v)}{\|r\|_V} \geq \frac{-b(z,v)}{\|z\|_V} = \frac{\left\langle \int_D v \operatorname{div} z\, dx \right\rangle}{\|z\|_V} = \frac{\|v\|_W^2}{\|z\|_V} \geq \beta \|v\|_W \quad v \in W, \tag{4.27}
$$

with the constant $\beta := \frac{1}{C_D}$. The above conditions (4.24)-(4.27) ensure the existence and uniqueness of the solution to problem (4.13). Inequalities (4.23) are then established by using the usual techniques for saddle point problems [1]. First, using (4.13), (4.17) and the Cauchy-schwarz inequality, we estimate

$$\begin{aligned}
\|u^{(M)}\|_W^2 &= \left\langle \left(u^{(M)}, u^{(M)} \right) \right\rangle, \\
&\le \frac{1}{\alpha_{\min}} a_M(u^{(M)}, u^{(M)}), \\
&= \frac{1}{\alpha_{\min}} \left\langle \left(f, p^{(M)} \right) \right\rangle.
\end{aligned}$$

Thus,

$$\|u^{(M)}\|_W^2 \le \frac{1}{\alpha_{\min}} \|f\| \|p^{(M)}\|_W. \tag{4.28}$$

Using the inf-sup condition and the continuity of $a_M(\cdot,\cdot)$ we obtain,

$$\|p^{(M)}\|_W \le \frac{1}{\beta} \sup_{r\in V} \frac{b(r, p^{(M)})}{\|r\|_V} = \frac{1}{\beta} \sup_{r\in V} \frac{-a_M(u^{(M)}, r)}{\|r\|_V} \le \frac{\alpha_{\max}}{\beta} \|u^{(M)}\|_W. \tag{4.29}$$

From (4.28),(4.29) we get,

$$\|u^{(M)}\|_W \le \frac{\alpha_{\max}}{\beta \alpha_{\min}} \|f\|. \tag{4.30}$$

Since,

$$\begin{aligned}
\|\operatorname{div} u^{(M)}\|_W^2 &= \left\langle \left(\operatorname{div} u^{(M)}, \operatorname{div} u^{(M)} \right) \right\rangle, \\
&= -b(u^{(M)}, \operatorname{div} u^{(M)}), \\
&= \left\langle \left(f, \operatorname{div} u^{(M)} \right) \right\rangle, \\
&\le \|f\| \|\operatorname{div} u^{(M)}\|_W.
\end{aligned}$$

we get,

$$\|\operatorname{div} u^{(M)}\|_W \leq \|f\|. \tag{4.31}$$

By using (4.30), (4.31) and $\|u^{(M)}\|_V^2 = \|u^{(M)}\|_W^2 + \|\operatorname{div} u^{(M)}\|_W^2$ we obtain that

$$\|u^{(M)}\|_V \leq \left(\left(\frac{\alpha_{\max}}{\beta \alpha_{\min}} \right)^2 + 1 \right)^{\frac{1}{2}} \|f\|. \tag{4.32}$$

From (4.29)-(4.30) we get,

$$\|p^{(M)}\|_W \leq \frac{\alpha_{\max}^2}{\alpha_{\min} \beta^2} \|f\|. \tag{4.33}$$

4.4 Galerkin approximation.

In this section we will construct Galerkin approximations of the solution $(u^{(M)}, p^{(M)})$ to problem (4.13) by building finite dimensional subspaces $V_{h\nu,M} \subset V$ and $W_{h\nu,M} \subset W$. To that end, we combine mixed finite element functions of $x \in D$ and 2^M-basis of $\{Z_k\}_{k=1}^M$ of S_M. In what follows, $h > 0$ and $\nu \geq 1$ will denote discretization parameters associated with the finite element approximation on D. Consider τ to be a family of uniform meshes Δ_h on D. Each mesh is a partition of D into rectangles K_j such that $\overline{D} = \cup_{j=1}^N \overline{K}_j$, $\overline{K}_i \cap \overline{K}_j$ $(i \neq j)$ is either empty or a common vertex or an entire edge. The parameter h denotes the maximal diameter of the mesh elements. We choose finite-dimensional subspaces of $H_0(\operatorname{div}, D)$ and $L^2(D)$ that are inf-sup stable for the deterministic discrete saddle point problem. Let $\mathbb{P}_\nu(K)$ denote the space of polynomials defined on K with degree $\leq \nu$. We use two families of elements RT and BDM. The corresponding spaces are denoted as follows [1, 49]

$$\mathbb{P}_{\nu}^{\mathrm{RT}}(K) = (\mathbb{P}_{\nu-1}(K))^d \oplus x\,\mathbb{P}_{\nu-1}(K); \quad \mathbb{P}_{\nu}^{\mathrm{BDM}}(K) = (\mathbb{P}_{\nu}(K))^d.$$

In this dissertation, we use the Raviart-Thomas element of the lowest order. In Chapter 5 we show a typical function of the Raviart-Thomas type of the lowest order. Then, we set

$$X_{h\nu}^{\mathrm{div}} := \{r \in H_0(\mathrm{div}, D);\, r|_K \in P_{\nu}(K)\ \forall K \in \Delta_h\}. \tag{4.34}$$

The compatible subspace of $L^2(D)$ is defined as follows [1]:

$$X_{h\nu}^{0} := \left\{v \in L^2(D);\, v|_K \in P_{\nu-1}(K)\ \forall K \in \Delta_h\right\}. \tag{4.35}$$

We note that

$$S_M := \mathrm{span}\left\{Z_k(\omega) : 1 \le k \le 2^M\right\} \subseteq L^2(\Omega, \mathcal{F}, P), \tag{4.36}$$

where Z_k are the basis function introduced in (4.11). Now, choosing

$$V_{h\nu,M} := X_{h\nu}^{\mathrm{div}} \otimes S_M \quad \text{and} \quad W_{h\nu,M} := X_{h\nu}^{0} \otimes S_M, \tag{4.37}$$

where $X \otimes Y$ is the tensor product of the spaces X and Y defined by

$$X \otimes Y = \sum_{i,j}^{m} x_1 y_j : \, x_i \in X, y_i \in Y, \; 1 \leq i, j \leq m.$$

This lead to the following (sGFEM) formulation for problem (4.13): find $(u_{h,\nu}^{(M)}, p_{h,\nu}^{(M)}) \in V_{h\nu,M} \times W_{h\nu,M}$ satisfying

$$\begin{aligned} a_M(u_{h,\nu}^{(M)}, r) + b(r, p_{h,\nu}^{(M)}) &= 0 \qquad && \forall r \in V_{h\nu,M}, \\ b(u_{h,\nu}^{(M)}, v) &= -\langle (f, v) \rangle \qquad && \forall v \in W_{h\nu,M}. \end{aligned} \tag{4.38}$$

To discuss the convergence and error analysis for the discrete form, we assume that $D \subset \mathbb{R}^2$. In this setting, lemma 4.3 can be restated as

Lemma 4.5 *For any $w \in L^2(D) \otimes S_M$ there exists a vector valued function $z \in H_0(\mathrm{div}, D) \otimes S_M$ such that $z \in H^\epsilon(D) \otimes S_M$ for some $\epsilon > 0$, $\mathrm{div} z = w$, and the following hold*

$$\|z\|_V \leq C_D \|w\|_W,$$

$$\|z\|_{L^2_P(\Omega; H^\epsilon(D))} \leq C_{\mathrm{reg}} \|w\|_W.$$

The constant $C_D > 0$ depends on the domain D, while the constant $C_{reg} > 0$ depends on D and ϵ.

Proof. The proof of this lemma is established in a similar way as the proof of Lemma 4.3 where in this case $w \in L^2(D) \otimes S_M$. We define $z = -\nabla u$, where $u \in H^1_{0,\mathrm{Dir}}(D) \otimes S_M$ satisfies (4.20) for any $v \in H^1_{0,\mathrm{Dir}}(D) \otimes S_M$ and follow the same prove of lemma 4.3.

Theorem 4.3 *Let $D \subset \mathbb{R}^2$. For any $h > 0$, $\nu \geq 1$, and $M \in \mathbb{N}_0$, the discrete problem (4.38) has a unique solution $(u_{h,\nu}^{(M)}, p_{h,\nu}^{(M)}) \in V_{h\nu,M} \times W_{h\nu,M}$ where, $\tilde{\beta} > 0$ is the discrete inf-sup constant of (4.38) and is independent of h,ν and M.*

Proof. From, the results of the general theory of saddle point problem we verify the conditions that ensure that the uniqueness and existence of the solution. First, the continuity of the bilinear forms is established as:

$$|a_M(u, r)| \leq \alpha_{\max} \|u\|_V \|r\|_V \qquad\qquad \forall\, u, r \in V_{h\nu,M}, \tag{4.39}$$

$$|b(r, v)| \leq \alpha_{\max} \|r\|_V \|v\|_W \quad \forall\, r \in V_{h\nu,M}, \quad \forall\, v \in W_{h\nu,M}. \tag{4.40}$$

We consider the discrete null-space associated with $b(\cdot, \cdot)$:

$$V_{h\nu,M}^0 := \{r \in V_{h\nu,M};\, b(r, v) = 0 \quad \forall v \in W_{h\nu,M}\}\,. \tag{4.41}$$

Observing that

$$\operatorname{div} r(x, \cdot) \in W_{h\nu,M} \quad \forall\, r \in V_{h\nu,M}, \tag{4.42}$$

we have

$$\|r\|_V^2 = \langle \|r(x, \cdot)\| \rangle \quad \forall\, r \in V_{h\nu,M}^0.$$

The coercivity of $a_M(\cdot,\cdot)$ on $V^0_{h\nu,M}$ follows by using of the lower bound in (4.17):

$$\begin{aligned} a_M(r,r) &= \left\langle \int_D A_M^{-1}(x,\cdot)\,|r(x,\cdot)|^2 dx \right\rangle \\ &\geq \alpha_{\min} \left\langle \|r(x,\cdot)\|^2 \right\rangle = \alpha_{\min} \|r(x,\cdot)\|_V^2, \ \ \forall\, r \in V^0_{h\nu,M}. \end{aligned} \tag{4.43}$$

Next, we established the discrete inf-sup condition. To do that, first for any $v \in W_{h\nu,M} \subset L^2(D) \otimes S_M$ Lemma 4.5 leads to finding $z \in (H^\epsilon(D) \cap H_0(div, D) \otimes S_M(\Omega))$ for some $\epsilon > 0$ such that $\operatorname{div} z = v$ and

$$\|z\|^2_{L^2_P(\Omega, H^\epsilon(D))} = \left\langle \|z\|^2_{H^\epsilon(D)} \right\rangle \leq C^2_{\text{reg}} \|v\|^2_W. \tag{4.44}$$

Now, we use the $H(\text{div}, D)$-conforming $h\nu$-interpolation operator defined by

$$Q_{\text{div},h\nu} : H^\epsilon(D) \cap H_0(\text{div}, D) \to X^{\text{div}}_{h\nu},$$

to define $z_{h\nu}(x,\cdot) := Q_{\text{div},h\nu} z(x,\cdot) \in X^{\text{div}}_{h\nu} \otimes S_M = V_{h\nu,M}$. Using the properties of $Q_{\text{div},h\nu}$ and using that $\operatorname{div} z = v$, we find

$$\operatorname{div} z_{h\nu} = \operatorname{div}(Q_{\text{div},h\nu} z) = Q_{0,h\nu} v = v \quad \forall\, v \in W_{h\nu,M}.$$

Using (4.44) the following estimate holds:

$$\begin{aligned} \|z_{h\nu}\|^2_V = \|z_{h\nu}\|^2_{L^2_P(\Omega, H(\text{div}, D))} &\leq C^2_{\text{int}} (\|z\|^2_{L^2_P(\Omega, H^\epsilon(D))} + \|\text{div} z\|^2_W) \\ &\leq C^2_{\text{int}} (1 + C^2_{\text{reg}}) \|v\|^2_W, \end{aligned} \tag{4.45}$$

where $C_{\text{int}} > 0$ is the stability constant which is independent of h and ν for $\varphi_{\text{div},h\nu}$. Thus, the inf-sup stability follows in the usual manner:

$$\sup_{r \in V_{h\nu,M}} \frac{b(r,v)}{\|r\|_V} \geq \frac{-b(z_{h\nu},v)}{\|z_{h\nu}\|_V} = \frac{\left\langle \int_D v \operatorname{div} z_{h\nu}\, dx \right\rangle}{\|z_{h\nu}\|_V} = \frac{\|v\|_W^2}{\|z_{h\nu}\|_V} \geq \tilde{\beta}\|v\|_W \quad v \in W, \tag{4.46}$$

where $\tilde{\beta} = \left(C_{\text{int}}\sqrt{1 + C_{\text{reg}}^2}\right)^{-1}$. The conditions (4.39 -4.40 - 4.43 and 4.46) ensure the existence and the uniqueness of the solution of (4.38).

4.5 Error Analysis.

In this section error bounds for the error analysis that has been introduced in each discretization steps are obtained. The first truncated we performed was in the truncated representation of $A^{-1}(x,\omega)$ in (4.10). Then (4.9) was replaced by perturbed problem (4.13). In Section 4.5.1 we obtain bounds for the corresponding perturbed error $\|u - u^{(M)}\|_V$ and $\|p - p^{(M)}\|_W$ in terms of the discretization parameter M. After approximating $(u^{(M)}, p^{(M)})$ by the sGFEM solution $(u_{h,\nu}^{(M)}, p_{h,\nu}^{(M)})$ we estimate the corresponding discretization error in Section 4.5.2.

4.5.1 Perturbation error.

The perturbation error is estimated using Strang's lemma in the following lemma.

Lemma 4.6 *Let $(u,p) \in V \times W$ be the solution of (4.9) and let $(u^{(M)}, p^{(M)}) \in$*

$\mathrm{V} \times \mathrm{W}$ *be the solution of (4.13). Then,*

$$\|u - u^{(M)}\|_{\mathrm{V}} \leq \frac{C_{\max}}{\beta C_{\min} \alpha_{\min}} \|f\| \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)}, \tag{4.47}$$

$$\|p - p^{(M)}\|_{\mathrm{W}} \leq \frac{1}{\beta^2} \frac{C_{\max}}{C_{\min}} \left(1 + \frac{\alpha_{\max}}{\alpha_{\min}}\right) \|f\| \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)}, \tag{4.48}$$

where the lower and upper bounds in (4.5) (resp. in (4.16)) are $C_{\min}$ *and* $C_{\max}$ *(resp.* $\alpha_{\min}$ *and* $\alpha_{\max}$*) and* β *is the inf-sup constant in Lemma 4.1.*

Proof. Let us put $e_u := u - u^{(M)} \in V$ and $e_p := p - p^{(M)} \in W$. Then from (4.9) and (4.13) we get,

$$b(e_u, v) = 0 \quad \forall\, v \in W \text{ implying that } \operatorname{div} e_u = 0 \quad a.e \text{ in } D \times \Omega,$$

$$a(u, e_u) = -b(e_u, p) = 0 \quad \text{and} \quad a_M(u^{(M)}, e_u) = b(e_u, p^{(M)}) = 0.$$

Using the definitions of $a(\cdot,\cdot)$ and $a_M(\cdot,\cdot)$ in (4.8) and (4.15) and recalling the Cauchy Schwarz inequality and the lower bound for $A^{-1}(x,\omega)$ in (4.16), we have

$$\begin{aligned}
\alpha_{\min}\|e_u\|_V^2 &= \alpha_{\min}\left\langle \|e_u\|^2 \right\rangle \leq a_M(e_u, e_u) \\
&= a_M(u - u^M, e_u) \\
&= a_M(u, e_u) - a_M(u^{(M)}, e_u) \\
&= a_M(u, e_u) - a_(u, e_u) \\
&= E\left[\int_D (A_M^{-1}(x,\cdot) - A^{-1}(x,\cdot))u(x,\cdot)\cdot e_u(x,\cdot)dx\right] \\
&\leq \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)}|E[(u, e_u)]| \\
&\leq \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)}(E[\|u\|^2])^{\frac{1}{2}}(E[\|e_u\|^2])^{\frac{1}{2}} \\
&= \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)}\|u\|_V\|e_u\|_V.
\end{aligned}$$

Therefore,

$$\|e_u\|_V \leq \frac{1}{\alpha_{\min}}\|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)}\|u\|_V. \tag{4.49}$$

Since $\mathrm{div} e_u = 0$, and combining (4.49) with the upper bound for $\|u\|_V$ from Lemma 4.1, we obtain

$$\|u - u^{(M)}\|_V \leq \frac{1}{\alpha_{\min}}\left(\left(\frac{C_{\max}}{\beta C_{\min}}\right)^2 + 1\right)^{\frac{1}{2}}\|f\|\|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)}. \tag{4.50}$$

In (4.9) we use inf-sup condition we obtain

$$\|e_p\|_W \leq \frac{1}{\beta}\sup_{r\in V}\frac{b(r, e_p)}{\|r\|_V}. \tag{4.51}$$

In order to estimate $b(r, e_p)$ for any $r \in V$, we again use the definition of $a(\cdot,\cdot)$ and $a_M(\cdot,\cdot)$ and the variational formulation (4.9) and (4.13). We have

$$
\begin{aligned}
|b(r, e_p)| &= b(r, p) - b(r, p^{(M)}) \\
&= -a(u, r) + a_M(u^{(M)}, r) \\
&= -a_M(e_u, r) - (a(u, r) - a_M(u, r)) \\
&= -E\left[\int_D A_M^{-1}(x,\cdot) e_u(x,\cdot) \cdot r(x,\cdot) dx\right] \\
&- E\left[\int_D (A^{-1}(x,\cdot) - A_M^{-1}(x,\cdot)) u(x,\cdot) \cdot r(x,\cdot) dx\right].
\end{aligned}
$$

Using the upper bound for $A_M^{-1}(x,\omega)$ in (4.16) and employing the Cauchy Schwarz inequality we obtain

$$
\begin{aligned}
|b(r, e_p)| &\le \alpha_{\max} \|e_u\|_W \|r\|_W \\
&+ \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)} \|u\|_W \|r\|_W \\
&\le \left(\alpha_{\max}\|e_u\|_V + \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)} \|u\|_W\right) \|r\|_V.
\end{aligned} \tag{4.52}
$$

From (4.52), (4.51) and (4.49) we obtain

$$
\|e_p\|_W \le \frac{1}{\beta}\left(\frac{\alpha_{\max}}{\alpha_{\min}} + 1\right) \|A^{-1} - A_M^{-1}\|_{L_P^\infty(D\times\Omega)} \|u\|_W. \tag{4.53}
$$

Finally, using the upper bound for $\|u\|_W$ from Lemma 4.1 we get

$$
\|p - p^{(M)}\|_W \le \frac{1}{\beta^2} \frac{C_{\max}}{C_{\min}} \left(1 + \frac{\alpha_{\max}}{\alpha_{\min}}\right) \|f\| \|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)}. \tag{4.54}
$$

Combining Lemma 4.6 and Lemma 4.1, we estimate the perturbation error in terms of M in the following corollary

Corollary 4.4 *Suppose that $A^{-1}(x,\omega)$ satisfies Assumptions 4.1 and 4.2 and is represented by the expansion in (4.10). Let $(u,p) \in V \times W$ and $(u^{(M)}, p^{(M)}) \in \mathrm{V} \times \mathrm{W}$ be the solutions of (4.9) and (4.13), respectively. Then under the assumptions in Lemma 4.2, the following error bounds hold for sufficiently large M:*

$$\|u - u^{(M)}\|_{\mathrm{V}} + \|p - p^{(M)}\|_{\mathrm{W}} \leq \boldsymbol{C}\|f\|, \tag{4.55}$$

where

$$\boldsymbol{C} = \left(\frac{1}{\alpha_{\min}} \left(\left(\frac{C_{\max}}{\beta C_{\min}} \right)^2 + 1 \right)^{\frac{1}{2}} + \frac{1}{\beta^2} \frac{C_{\max}}{C_{\min}} \left(1 + \frac{\alpha_{\max}}{\alpha_{\min}} \right) \right) \epsilon_M$$

4.5.2 Stochastic Galerkin error.

We now consider the formulation in (4.13) and the sGFEM approximation (4.38). The objective is to obtain a bound for the approximation

$$E_{h\nu,M} := \|u^{(M)} - u_{h,\nu}^{(M)}\|_V + \|p^{(M)} - p_{h,\nu}^{(M)}\|_W \tag{4.56}$$

Lemma 4.7 *[21]. Let $(u^{(M)}, p^{(M)}) \in V \times W$ and $(u_{h,\nu}^{(M)}, p_{h,\nu}^{(M)}) \in V_{h\nu,M} \times W_{h\nu,M}$ be, respectively, solutions of problems (4.13) and (4.38). Assume that the inf-sup condition*

$$\inf_{v \in W_{h\nu,M}} \sup_{r \in V_{h\nu,M}} \frac{b(u_{h,\nu}^{(M)}, v)}{\|r\|_{V_{h\nu,M}} \|v\|_{W_{h\nu,M}}} \geq \hat{\beta} > 0, \tag{4.57}$$

is satisfied and let $a_M(\cdot,\cdot)$ *be uniformly coercive on the null-space of* $b(\cdot,\cdot)$*, that is, there exists* $\alpha_{\min} > 0$ *such that*

$$a_M(r,r) \geq \alpha_{\min} \|r\|_V^2 \ \ \forall\, r \in V^0_{h\nu,M}. \tag{4.58}$$

Then one has the following estimate, with constant c *depending of* $\|a\|$*,* $\|b\|$*,* $\hat{\beta}$*,* $\alpha_{\min}$ *but independent of* h*:*

$$\|u^{(M)} - u^{(M)}_{h,\nu}\|_V + \|p^{(M)} - p^{(M)}_{h,\nu}\|_W \leq C_1 \left(\inf_{r \in V_{h\nu,M}} \|u^{(M)} - r\|_V + \inf_{v \in W_{h\nu,M}} \|p^{(M)} - v\|_W \right. \tag{4.59}$$

The proof of this lemma can be found in [21].

Now, we introduce the following orthogonal projections

$$\Pi^{\mathrm{div},\perp}_{h\nu} : H_0(\mathrm{div}, D) \to X^{\mathrm{div}}_{h\nu}, \quad \Pi^0_{h\nu} : L^2(D) \to X^0_{h\nu},$$

with the corresponding inner products in $H_0(\mathrm{div}, D)$, $L^2(D)$ respectively such that for any $u \in H_0(\mathrm{div}, D)$ the projection $\Pi^{\mathrm{div},\perp}_{h\nu}$ satisfies

$$(u - \Pi^{\mathrm{div},\perp}_{h\nu} u, r)_{H(\mathrm{div},D)} = 0 \quad \forall r \in X^{\mathrm{div}}_{h\nu},$$

and for any $p \in L^2(D)$ the projection $\Pi^0_{h\nu}$ satisfies

$$(p - \Pi^0_{h\nu} p, v)_{L^2(D)} = 0 \qquad \forall v \in X^0_{h\nu}.$$

Where $(\cdot,\cdot)_{H(\text{div},D)}$ and $(\cdot,\cdot)_{L^2(D)}$ denotes the inner product in $H(\text{div},D)$, $L^2(D)$ respectively.

Lemma 4.8 *[1] Let $D \subset \mathbb{R}^2$ and let $(u^{(M)}, p^{(M)}) \in L^2_P(\Omega, H^\epsilon(\text{div}, D)) \times L^2_P(\Omega, H^\epsilon(D))$ $(\epsilon > 0)$ be the solution to problem (4.13). Then following holds*

$$\begin{aligned}
&\inf_{r\in V_{h\nu,M}} \|u^{(M)} - r\|_V \\
&\qquad + \inf_{r\in W_{h\nu,M}} \|p^{(M)} - v\|_V \\
&\qquad \le \|u^{(M)} - \Pi^{\text{div},\perp}_{h\nu} u^{(M)}\|_V + \|p^{(M)} - \Pi^0_{h\nu} p^{(M)}\|_W \\
&\qquad \le C_2 h^{min\{\epsilon,\nu\}} \nu^{-\epsilon} \left(\|u^{(M)}\|_{L^2_P(\Omega,H^\epsilon(\text{div},D))} + \|p^{(M)}\|_{L^2_P(\Omega,H^\epsilon(D))} \right)
\end{aligned}$$

The next theorem is consequence of Lemma 4.7 and Lemma 4.8.

Theorem 4.5 *There exists a positive constant C depending on the constants $\alpha_{\min}$ and $\alpha_{\max}$ in (4.16) and on the discrete inf-sup constant β but independent of the discretization parameters (h,ν) such that*

$$\|u^{(M)} - u^{(M)}_{h,\nu}\|_{\text{V}} + \|p^{(M)} - p^{(M)}_{h,\nu}\|_{\text{W}} \le C h^{min\{\epsilon,\nu\}} \nu^{-\epsilon} \left(\|u^{(M)}\|_{L^2_P(\Omega,H^\epsilon(\text{div},D))} + \|p^{(M)}\|_{L^2_P(\Omega,H^\epsilon(D))} \right) \tag{4.60}$$

Proof. From lemmas 4.7 and 4.8 we get

$$\|u^{(M)} - u^{(M)}_{h,\nu}\|_V + \|p^{(M)} - p^{(M)}_{h,\nu}\|_W \le C h^{min\{\epsilon,\nu\}} \nu^{-\epsilon} \left(\|u^{(M)}\|_{L^2_P(\Omega,H^\epsilon(\text{div},D))} + \|p^{(M)}\|_{L^2_P(\Omega,H^\epsilon(D))} \right), \tag{4.61}$$

where,

$$C = C_1 C_2$$

CHAPTER 5

EFFICIENT BLOCK PRECONDITIONERS FOR SECOND-ORDER ELLIPTIC PROBLEMS

The organization of this Chapter is as follows: Section 5.1 introduces an introduction of the model and the preconditioning which are used. In Sections 5.2 and 5.3, we introduce some spaces and their norms and Raviart-Thomas element with lowest degree. In Sections 5.4 and 5.5, we describe the variational formulation of problem (5.1) and state its well-posedness and define the finite element matrices generated from this variational formulation. Eigenvalues bounds of the preconditioned matrix $P_{\alpha_1,\alpha_2}^{-1}\hat{A}$ are discussed and established in Section 5.6. In this section, we show that the eigenvalues are contained in a union of two intervals $(-a,-b] \cup [1]$ where the first interval of small size. Different types of preconditioners will be discussed in Section 5.7. Restarted GMERS method will be studied in Section 5.8. Several numerical experiments that show the efficiency of our preconditioning techniques are discussed in Section 5.9.

5.1 Introduction

We consider a fluid flow model in porous media,

$$\begin{aligned} A^{-1}\vec{u} - \nabla p &= 0, && \text{in } D \\ \nabla \cdot \vec{u} &= -f, && \text{in } D \\ p &= g, && \text{on } \partial D_{Dir} \\ \vec{n} \cdot \vec{u} &= 0, && \text{on } \partial D_{Neu} \end{aligned} \tag{5.1}$$

where, ∂D_{Dir} and ∂D_{Neu} are nonempty set, $A^{-1} = A^{-1}(x, y)$ is 2×2 bounded, symmetric, and uniformly positive definite matrix-valued function, p and $\vec{u}$ the pressure and velocity solutions, f and g are functions of two variables (x, y) . We assume that positive constants γ and Γ exist such that $0 < \gamma \leq \Gamma$ and

$$\gamma(\vec{v}, \vec{v}) \leq (A^{-1}\vec{v}, \vec{v}) \leq \Gamma(\vec{v}, \vec{v}),$$

for every $\vec{v} : D \longrightarrow \mathbb{R}^2$ where, $(\cdot, \cdot)$ is the inner product defined in Section 5.2. After applying Raviart-Thomas mixed finite element, the problem (5.1) gives [52] an indefinite linear system of the form

$$\underbrace{\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} \underline{u} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \vec{g} \\ -\vec{f} \end{bmatrix}, \tag{5.2}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite, $B \in \mathbb{R}^{m\times n}$, $\vec{f} \in \mathbb{R}^{n\times 1}$, $\vec{g} \in \mathbb{R}^{m\times 1}$, $\underline{u}$ represent the discrete velocity and $\underline{p}$ represent the discrete pressure solutions. The mixed finite element approximation of Darcy's equation yields a huge linear system with very large condition number. It means that any iterative method for the linear system is very slow. Powell and Silvester provide block diagonal preconditioning technique in [9], with the optimal preconditioning in the following form

$$P_{div} = \begin{bmatrix} A+D & 0 \\ 0 & N \end{bmatrix}, \tag{5.3}$$

where, D and N are cheap to assemble, N is a diagonal matrix. In this chapter, we evaluate the general block diagonal and block triangular preconditioning with the parameters α_1 and α_2 in the form

$$P_{\alpha_1,\alpha_2} = \begin{bmatrix} A+\alpha_1 D & 0 \\ (1-\alpha_1\alpha_2)B & \alpha_2 N \end{bmatrix}. \tag{5.4}$$

Observe that $P_{1,1} = P_{div}$ and P_{α_1,α_2} is block diagonal matrix. If $\alpha_1, \alpha_2 \neq 1$ then the preconditioner P_{α_1,α_2} is block triangular matrix.

5.2 Notation

Let $D \subseteq \mathbb{R}^2$ be a convex polygon with boundary $\partial D_{Dir} \cup \partial D_{Neu}$. $L^2(D)$ is the space of square integrable functions with inner product $(\cdot, \cdot)$. Define the subspace,

$$H(\mathrm{div}; D) := \{\vec{v} \in L^2(D)^2 \mid \nabla \cdot \vec{v} \in L^2(D)\}.$$

The inner product in this subspace is given by

$$(\vec{u}, \vec{v})_{div} = (\vec{u}, \vec{v}) + (\nabla \cdot \vec{u}, \nabla \cdot \vec{v}).$$

The associated norm is defined by $\| \cdot \|_{div}$ also, we define the sobolev space,

$$H^1(D) := \{w \in L^2(D) \mid \nabla w \in L^2(D)^2\},$$

where

$$(w, v)_1 = (w, v) + (\nabla w, \nabla v),$$

and the corresponding norm is denoted by $\| \cdot \|_1$.

We define $H^{1/2}(\partial D_{Dir})$ to be the set of traces of $H^1(D)$ functions on ∂D_{Dir}.

5.3 Raviart-Thomas Finite Elements (RT0)

Assume that D is a rectangular partition $\{K_1, K_2, ..., K_s\}$ having maximum edge size h. Let $\{\tau_{h_1}, \tau_{h_2}, ...\}$ be a family of shape-regular-uniform partitions of D, where τ_h denotes a partition of D into a mesh of rectangles. Let τ_h be the associated Raviart-Thomas spaces $V_h \subset H(\text{div}, D)$ and $W_h \subset L^2(D)$ of index zero k [15, 64] are

$$V_h := \left\{ \vec{v}_h \in H(\text{div}, D) : \vec{v}_h|_K = \begin{pmatrix} c_1 + c_2 x \\ c_3 + c_4 y \end{pmatrix} \right\},$$

and

$$W_h := \{w_h \in L^2(D) : w_h|_K = w_1\},$$

where, $\vec{v}_h$ and w_h the velocity and pressure test functions respectively, and $c_1, c_2, c_3,\ c_4, w_1$ are constants.

5.4 Weak Formulation

We known from [14, 15] that (5.1) can be expressed as a saddle point problem (5.2) in two different function spaces.

$$V = \{\vec{v} \in H(\text{div}, D) \mid \vec{v} \cdot \vec{n} \ = 0 \quad \text{on} \quad \partial D_{Neu}\},$$

and $W = L^2(D)$. Multiplying by $\vec{v} \in V$ and $w \in W$ in (5.1): find $(\vec{u}, p) \in V \times W$ satisfying

$$
\begin{aligned}
(A^{-1}\vec{u}, \vec{v}) + (p, \nabla \cdot \vec{v}) &= \langle g, \vec{v} \cdot \vec{n} \rangle, \qquad \forall v \in V \\
(w, \nabla \cdot \vec{u}) &= -(f, w), \qquad \forall w \in W
\end{aligned}
\tag{5.5}
$$

where, $\langle g, \vec{v} \cdot \vec{n} \rangle = \int_{\partial \Omega_D} g \vec{v} \cdot \vec{n} ds$. Since, $(A^{-1}\vec{u}, \vec{v})$ is bounded, the stability theory of [21, 34, 52], shows that a solution $(\vec{u}, p)$ exists and unique if and only if there exist $\alpha_0 > 0$ and $\beta_0 > 0$ satisfying

$$
(A^{-1}\vec{v}, \vec{v}) \geq \alpha_0 \parallel \vec{v} \parallel_{div}^2 , \qquad \forall v \in Z
$$

$$
\sup_{\vec{v} \in V \setminus \{\vec{0}\}} \frac{(w, \nabla \cdot \vec{v})}{\|\vec{v}\|_{div}} \geq \beta_0 \|w\|_0 , \qquad \forall w \in W
$$

where $Z = \{\vec{v} \in V | (w, \nabla \cdot \vec{v}) = 0 \quad \forall w \in W\}$.

5.5 Finite Element Matrices

By using Raviart-Thomas approximation, we start by choosing finite dimensional subspaces $V_h \subset V$ and $W_h \subset W$. Then the problem becomes find: $(u_h, p_h) \in V_h \times W_h$, satisfying,

$$
\begin{aligned}
(A^{-1}\vec{u}_h, \vec{v}_h) + (\nabla \cdot \vec{v}_h, p_h) &= \langle g, \vec{v}_h \cdot \vec{n} \rangle , \qquad \forall \vec{v}_h \in V_h \\
(w_h, \nabla \cdot \vec{u}_h) &= -(f, w_h) . \qquad \forall w_h \in W_h
\end{aligned}
\tag{5.6}
$$

Stability result can be found in [49, 52] . Now, let $V_h = \text{span}\{\vec{\varphi}_i\}_{i=1}^n$ and $W_h = \text{span}\{\phi_j\}_{j=1}^m$. In (5.2) the finite element matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ are defined by,

$$A_{i,j} = (A^{-1}\vec{\varphi}_i, \vec{\varphi}_j) \text{ for } \quad 1 \le i,j \le n \text{ and } \quad B_{k,j} = (\phi_k, \nabla\cdot\vec{\varphi}_j), \quad 1 \le k \le m, 1 \le j \le n,$$

and the vectors $\underline{g} \in \mathbb{R}^n$ and $\underline{f} \in \mathbb{R}^m$ by $g_i = \langle g, \vec{\varphi}_i \cdot \vec{n}\rangle$ and $f_k = -(f, \varphi_k)$ the approximate solution is $\vec{u_h} = \Sigma_{i=1}^n u_i \vec{\varphi}_i$ and $p_h = \Sigma_{j=1}^m p_j \phi_j$, where u_i, p_j are the components of $\vec{u_h}$, $\vec{p_h}$, respectively. To develop our preconditioner, we define the velocity mass matrix $M \in \mathbb{R}^{n\times n}$, the velocity divergence matrix $D \in \mathbb{R}^{n\times n}$, and the pressure mass matrix $N \in \mathbb{R}^{m\times m}$ by

$$\begin{aligned}
M_{i,j} &= (\vec{\varphi}_i, \vec{\varphi}_j), & 1 \le i,j \le n \\
D_{i,j} &= (\nabla\cdot\vec{\varphi}_i, \nabla\cdot\vec{\varphi}_j), & 1 \le i,j \le n \\
N_{r,s} &= (\varphi_r, \varphi_s), & 1 \le r,s \le m
\end{aligned} \tag{5.7}$$

which yields discrete norms $\|\cdot\|_{0,v_h}$ and $\|\cdot\|_{div}$ on V_h and $\|\cdot\|_{0,w_h}$ on W_h, then

$$\|\vec{v}_h\|_{0,N}^2 = \underline{v}^T M \underline{v}, \quad \|\vec{v_h}\|_{div}^2 = \underline{v}^T (M + D)\underline{v}, \quad \|w_h\|_{0,M}^2 = \underline{w}^T N \underline{w}\,, \tag{5.8}$$

where $\underline{v}$, $\underline{w}$ are the vectors of coefficients related to $\vec{v_h}$ and w_h. Now the Brezzi's discrete inf-sup stability condition is

$$\beta^2 \le \frac{\underline{w}^T (M+D)^{-1} B^T \underline{w}}{\underline{w}^T N \underline{w}}\,, \qquad \forall\, \underline{w} \in \mathbb{R}^m \backslash \{\underline{0}\} \tag{5.9}$$

It can be shown from [9],

$$\frac{\underline{w}^T(M+D)^{-1}B^T\underline{w}}{\underline{w}^T N\underline{w}} \le 1\,. \qquad \forall\,\underline{w}\in\mathbb{R}^m\backslash\{\underline{0}\} \tag{5.10}$$

5.6 Eigenvalues

In this Section, we will show that the eigenvalues of the preconditioned matrix $P^{-1}_{\alpha_1,\alpha_2}\hat{A}$ are contained in $(-a,-b]\cup[1]$ where the first interval is of small length, and a, b are defined in (5.12). This result is summarized in Theorem 6.1. Before we state the theorem we need to state the following lemmas

Lemma 5.1 [9] *If $\nabla\cdot V_h\in W_h$, then $D=B^TN^{-1}B$.*

The proof of this lemma can be found in [9]. The following lemma states that the minimum eigenvalue $\mu_{\min}$ of the Schur complement matrix $BA^{-1}B^T$ can be expressed in terms of the mesh size h.

Lemma 5.2 [9] *Let $\mu_{\min}$ denotes the minimum eigenvalue of $BA^{-1}B^T$. There exists a constant C, independent of h, such that $\mu_{\min}\ge Ch^2$.*

The proof of this lemma can be found in [9]. The next theorem will state the clustering behavior of the preconditioned matrix $P^{-1}_{\alpha_1,\alpha_2}\hat{A}$.

Theorem 5.1 *If τ_h is a quasi-uniform mesh, then the $n+m$ eigenvalues of*

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} \underline{u} \\ \underline{p} \end{bmatrix} = \lambda \begin{bmatrix} A+\alpha_1 D & 0 \\ (1-\alpha_1\alpha_2)B & \alpha_2 N \end{bmatrix}\begin{bmatrix} \underline{u} \\ \underline{p} \end{bmatrix}, \tag{5.11}$$

lie in the set,

$$\left(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}\left(\frac{c\,\mu_{\min}}{|K|_{\min}+\alpha_1\mu_{\min}}\right)\right] \cup \{1\}\,, \qquad (5.12)$$

where $\mu_{\min}$ *is the minimum eigenvalue of* $BA^{-1}B^T$, $|K|_{min}$ *is the area of the smallest rectangle* τ_h, *and* $c > 0$.

proof. The eigenvalues $\{\lambda_i\}_{i=1}^{m+n}$ of (5.11) satisfy

$$A\underline{u} + B^T\underline{p} = \lambda A\underline{u} + \alpha_1\lambda D\underline{u}\,,$$

$$B\underline{u} = (1-\alpha_1\alpha_2)\lambda B\underline{u} + \alpha_2\lambda N\underline{p}\,.$$

We first show that $\lambda = 1$ is an eigenvalue of (5.11) of geometric multiplicity n. Let $\underline{u} \in \mathbb{R}^n$ be any nontrivial vector and let $\underline{p} = \alpha_1 N^{-1}B\underline{u}$, then

$$A\underline{u} + B^T\underline{p} = A\underline{u} + B^T\alpha_1 N^{-1}B\underline{u} = A\underline{u} + \alpha_1 D\underline{u}\,.$$

Moreover,

$$(1-\alpha_1\alpha_2)B\underline{u} + \alpha_2 N\underline{p} = B\underline{u} - \alpha_1\alpha_2 B\underline{u} + \alpha_1\alpha_2 NN^{-1}B\underline{u} = B\underline{u}\,.$$

This shows that $\lambda = 1$ is an eigenvalue of geometric multiplicity at least n. Now assume that (5.11) has $k > n$ eigenvectors corresponding to the eigenvalue $\lambda = 1$. Then there are vectors $\begin{bmatrix} u_i \\ p_i \end{bmatrix}$, $1 \le i \le k$ satisfying

$$B^T\underline{p}_i = \alpha_1 B^T N^{-1}B\underline{u}_i, \qquad 1 \le i \le k$$

since $k > n$, there are scalars $\epsilon_1, \epsilon_2, ..., \epsilon_k$ such that $\sum_{i=1}^{k} \epsilon_i \underline{u}_i = 0$. Let $\tilde{p} = \sum_i^k \epsilon_i \underline{p}_i$ if $\tilde{p} = 0$, then the set $\begin{bmatrix} u_i \\ p_i \end{bmatrix}$, $1 \le i \le k$ are linearly independent. If $\tilde{p} \ne 0$, then

$$B^T \tilde{p} = \sum_{i=1}^{k} \epsilon_i B^T p_i = \alpha_1 B^T N^{-1} B \sum_{i=1}^{k} \epsilon_i \underline{u}_i = 0,$$

then $BA^{-1}B^T\tilde{p} = 0$, which contradicts the result of Lemma 5.2. Next, if $\lambda \ne 1$ is an eigenvalues, then

$$B^T \underline{p} = (\lambda - 1)A\underline{u} + \alpha_1 \lambda D \underline{u}\,, \tag{5.13}$$

$$(1 - \lambda + \lambda\alpha_1\alpha_2)B\underline{u} = \alpha_2 \lambda N \underline{p}\,. \tag{5.14}$$

(5.13) can be rewritten as,

$$B^T \underline{p} = (\lambda - 1)(A + \alpha_1 D)\underline{u} + \alpha_1 D \underline{u}\,. \tag{5.15}$$

Multiply (5.15) by $B(A + \alpha_1 D)^{-1}$ and use Lemma 5.1

$$B(A + \alpha_1 D)^{-1} B^T \underline{p} = (\lambda - 1)B\underline{u} + \alpha_1 B(A + \alpha_1 D)^{-1} B^T N^{-1} B \underline{u}\,. \tag{5.16}$$

(5.16) and (5.14) yield

$$(1 - \lambda + \alpha_1\alpha_2\lambda)B(A + \alpha_1 D)^{-1}B^T\underline{p} = \alpha_2\lambda(\lambda - 1)N\underline{p} + \alpha_1\alpha_2\lambda B(A + \alpha_1 D)^{-1}B^T\underline{p}\,.$$

The remaining m eigenvalue $\{\lambda_i\}_{i=1}^m$ satisfy

$$B(A+\alpha_1 D)^{-1}B^T\underline{p} = -\alpha_2\lambda N\underline{p}\,. \tag{5.17}$$

Since, $D = B^T N^{-1} B$, these m eigenvalues are the eigenvalues of the following matrix

$$-\frac{1}{\alpha_2}N^{\frac{-1}{2}}B(A+\alpha_1 B^T N^{-1}B)^{-1}B^T N^{\frac{-1}{2}}\,.$$

Rearranging gives,

$$-\frac{1}{\alpha_2}N^{\frac{-1}{2}}B(A+\alpha_1 B^T N^{-1}B)^{-1}B^T N^{\frac{-1}{2}} =$$

$$-\frac{1}{\alpha_2}N^{\frac{-1}{2}}BA^{\frac{-1}{2}}(I+\alpha_1 A^{\frac{-1}{2}}B^T N^{\frac{-1}{2}}N^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{-1}A^{\frac{-1}{2}}B^T N^{\frac{-1}{2}}.$$

Thus,

$$\frac{1}{\alpha_2}N^{\frac{-1}{2}}B(A+\alpha_1 B^T N^{-1}B)^{-1}B^T N^{\frac{-1}{2}} = \frac{1}{\alpha_2}X(I+\alpha_1 X^T X)X^T\,,$$

where, $X = N^{\frac{-1}{2}}BA^{\frac{-1}{2}}$. We now apply the Shermen-Morrison-Woodbury formula [25] to obtain

$$(I+\alpha_1 X^T X)^{-1} = I - \alpha_1 X^T(I+\alpha_1 XX^T)^{-1}X\,,$$

and so

$$\frac{1}{\alpha_2}X(I+\alpha_1 X^TX)^{-1}X^T = \frac{1}{\alpha_2}X(I-\alpha_1 X^T(I+\alpha_1 XX^T)^{-1}X)X^T\,.$$

Now, we use Lemma 3.1 of [59] with $X = N^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ to relate the eigenvalues of (5.17) to those of $BA^{-1}B^T$. Let, x_i be an eigenvector of XX^T and σ_i denote the corresponding eigenvalue. Then,

$$\begin{aligned}\frac{1}{\alpha_2}X(I+\alpha_1 X^TX)^{-1}X^Tx_i &= \frac{1}{\alpha_2}(XX^Tx_i - \alpha_1 XX^T(I+\alpha_1 XX^T)^{-1}XX^Tx_i)\\ &= \frac{1}{\alpha_2}\left(\sigma_i x_i - \alpha_1 XX^T(I+\alpha_1 XX^T)^{-1}XX^Tx_i\right)\,.\end{aligned} \tag{5.18}$$

Since,

$$\begin{aligned}(I+\alpha_1 XX^T)(XX^Tx_i) &= (I+\alpha_1 X^TX)(\sigma_i x_i)\,,\\ &= \sigma_i(x_i+\alpha_1\sigma_i x_i)\,,\\ &= (1+\alpha_1\sigma_i)\sigma_i x_i\,,\\ &= (1+\alpha_1\sigma_i)XX^Tx_i\,,\end{aligned}$$

implying that,

$$(I+\alpha_1 XX^T)^{-1}XX^Tx_i = \frac{1}{1+\alpha_1\sigma_i}\sigma_i x_i\,.$$

Thus,

$$\alpha_1 XX^T(I+\alpha_1 XX^T)^{-1}XX^Tx_i = \left(\frac{\alpha_1\sigma_i^2}{1+\alpha_1\sigma_i}\right)x_i\,. \tag{5.19}$$

From (5.18) and (5.19), we have

$$\frac{1}{\alpha_2} X(I + \alpha_1 X^T X)^{-1} X^T x_i = \frac{1}{\alpha_2} \left(\frac{\sigma_i}{1 + \alpha_1 \sigma_i} \right) x_i \, .$$

Hence, the eigenvalues of $\frac{1}{\alpha_2} X(I + \alpha_1 X^T X)^{-1} X^T$ are $\{\frac{1}{\alpha_2} \left(\frac{\sigma_i}{1+\alpha_1 \sigma_i} \right)\}_{i=1}^m$, where $\{\sigma_i\}_{i=1}^m$ are the positive eigenvalues of $XX^T = N^{\frac{-1}{2}} BA^{-1} B^T N^{\frac{-1}{2}}$. Since, $N^{-1} BA^{-1} B^T$ has the same eigenvalues as $N^{\frac{-1}{2}} BA^{-1} B^T N^{\frac{-1}{2}}$, the negative eigenvalues of our problem inside the interval

$$\left[-\frac{1}{\alpha_2} \max_i \frac{\sigma_i}{1 + \alpha_1 \sigma_i}, -\frac{1}{\alpha_2} \min_i \frac{\sigma_i}{1 + \alpha_1 \sigma_i} \right],$$

which can be written as

$$\left[-\frac{1}{\alpha_2} \frac{\sigma_{\max}}{1 + \alpha_1 \sigma_{\max}}, -\frac{1}{\alpha_2} \frac{\sigma_{\min}}{1 + \alpha_1 \sigma_{\min}} \right],$$

where $\sigma_{\min} = \min_i \{\sigma_i\}$, $\sigma_{\max} = \max_i \{\sigma_i\}$.

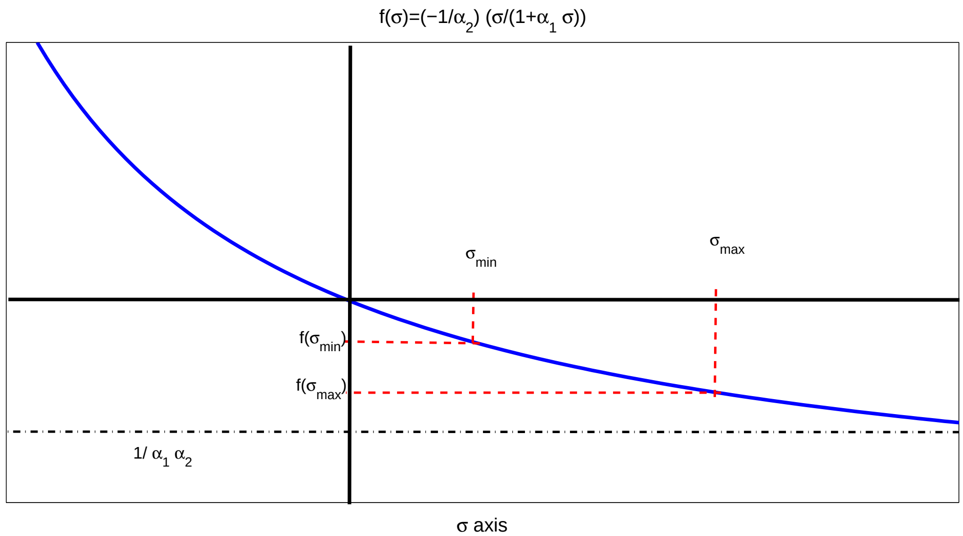

Figure 5.1: The negative eigenvalues of the matrix $X(I + \alpha_1 X^T X)^{-1} X^T$.

Since, the function $f(\sigma) = -\frac{1}{\alpha_2}\frac{\sigma}{1+\alpha_1\sigma}$ is decreasing in the interval from $-\frac{1}{\alpha_1\alpha_2}$ to ∞, figure 5.1 shows that if $\sigma \in [\sigma_{\min}, \sigma_{\max}]$, then $f(\sigma) \in [f(\sigma_{\max}), f(\sigma_{\min})]$. Also, from figure. 6.1,

$$-\frac{1}{\alpha_1\alpha_2} < -\frac{1}{\alpha_2}\frac{\sigma_{\min}}{1+\alpha_1\sigma_{\min}}\,.$$

Thus,

$$\{\lambda_i\}_{i=1}^m \in \left(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\sigma_{\min}}{1+\alpha_1\sigma_{\min}})\right]\,.$$

Finally, notice that the eigenvalues of the matrix N are the values $|K_1|, ..., |K_n|$. Then,

$$\frac{\mu_{\min}}{|K|_{\max}} \leq \sigma_{\min} \leq \frac{\mu_{\min}}{|K|_{\min}}\,, \tag{5.20}$$

where, $|K|_{\min}$ and $|K|_{\max}$ are the smallest and largest eigenvalues of N. Therefore,

$$1 + \alpha_1 \sigma_{\min} \le \frac{\alpha_1 \mu_{\min}}{|K|_{\min}} + 1\,,$$

$$-\frac{1}{1+\alpha_1\sigma_{\min}} \le -\frac{|K|_{\min}}{|K|_{\min} + \alpha_1 \mu_{\min}}\,. \tag{5.21}$$

From (5.20) and (5.21)

$$-\frac{\sigma_{\min}}{1+\alpha_1\sigma_{\min}} \le -\frac{c\mu_{\min}}{|K|_{\min} + \alpha_1 \mu_{\min}}\,,$$

where c is a constant independent of h, since the partition is quasi uniform. Therefore,

$$\{\lambda_i\}_{i=1}^{m} \in \left(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}\left(\frac{c\,\mu_{\min}}{|K|_{\min} + \alpha_1\mu_{\min}}\right)\right]. \tag{5.22}$$

This finishes the proof of the theorem. □

The following Corollary shows that the eigenvalues of the transpose preconditioned matrix $P^T_{\alpha_1,\alpha_2}\hat{A}$ lies in the same set as in (5.12).

Corollary 5.2 *If τ_h is a quasi-uniform mesh, the $n+m$ eigenvalues of*

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{p} \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} A + \alpha_1 D & (1-\alpha_1\alpha_2)B^T \\ 0 & \alpha_2 N \end{bmatrix}}_{P^T_{\alpha_1,\alpha_2}} \begin{bmatrix} \underline{u} \\ \underline{p} \end{bmatrix}, \tag{5.23}$$

lie in the set,

$$\left(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}\left(\frac{c\mu_{\min}}{|K|_{\min} + \alpha_1\mu_{\min}}\right)\right] \cup \{1\}\,, \tag{5.24}$$

where $\mu_{\min}$ is the minimum eigenvalue of $BA^{-1}B^T$, $|K|_{min}$ is the area of the smallest rectangle τ_h, and $c > 0$.

proof. Since, $(P^{-1}_{\alpha_1,\alpha_2}\hat{A})^T = \hat{A}^T(P^{-1}_{\alpha_1,\alpha_2})^T = \hat{A}P^{-T}_{\alpha_1,\alpha_2}$, thus the eigenvalues of (5.23) is the same as the eigenvalues of (5.11).

5.7 The Choice of the Parameters α_1, α_2

The eigenvalue bounds for the matrix (5.2) are obtained by Rusten and Winther in [68] and Silvester and Wathen in [17]. Different values for α_1 and α_2 lead to either block diagonal preconditioners or block tridiagonal preconditioners. The best choices of α_1, α_2 occur, for any $\epsilon > 0$ when $c = 1$ (recall that $c = \frac{|K|_{\min}}{|K|_{\max}} \leq 1$). We have two cases to consider. In the first case, when $\alpha_1\alpha_2 = 1$, the preconditioned matrix $P^{-1}_{\alpha_1,\alpha_2}\hat{A}$ is symmetric. Hence, MINRES method can be used to solve the linear system. But the question is how to pick values for the parameters α_1 and α_2. Equation (5.12) gives us that the length of the interval ϵ (clustering of the eigenvalue) is

$$\epsilon = -\frac{c\mu_{\min}}{\alpha_2|K|_{\min} + \mu_{\min}} + 1. \tag{5.25}$$

The smaller the value of ϵ the more clustering behavior of the eigenvalues. Solving for α_2 in equation (5.25) gives

$$\alpha_2 = \left(-1 + \frac{c}{1-\epsilon}\right)\frac{\mu_{\min}}{|K|_{\min}}. \tag{5.26}$$

In the second case when $\alpha_1\alpha_2 \neq 1$, the preconditioned matrix $P^{-1}_{\alpha_1,\alpha_2}\hat{A}$ is nonsymmetric. Then, GMRES method can be used to solve the linear system. From equation (5.12), the length of the interval ϵ is given by

$$\epsilon = \frac{1}{\alpha_2}\left(\frac{1}{\alpha_1} - \frac{c\mu_{\min}}{|K|_{\min} + \alpha_1\mu_{\min}}\right). \tag{5.27}$$

The larger the value of α_2 the smaller the length of the interval ϵ and the more clustering behavior of the eigenvalues. Solving for α_1 in equation (5.27) gives

$$\alpha_1 = \frac{-\epsilon|K|_{\min}\alpha_2 - (c-1)\mu_{\min} + \sqrt{\Delta}}{2\epsilon\mu_{\min}\alpha_2}, \tag{5.28}$$

where

$$\Delta = \epsilon^2|K|^2_{\min}\alpha_2^2 + 2(c+2)\mu_{\min}\epsilon\alpha_2|K|_{\min} + (c-1)^2\mu^2_{\min}.$$

5.8 Preconditioning

In computations, the matrices D and N are cheap to construct. The reason is that all entries of these matrices are constants and do not involve any integrations. A is symmetric positive definite, $N := \text{diag}[|K_1|, |K_2|, ..., |K_m|]$ where $|K_i|$ denotes the area of the ith finite element. α_1, α_2 are constants.

5.8.1 Restarted GMRES

The computational work and storage of GMRES grow Like $O(kn)$ [27]. For large values of k, the cost of GMRES in operations and storage may be prohibitive.

In such situations, k-step restarted GMRES or GMRES(k) is employed. After k GMRES steps, the GMRES iteration is restarted with a new current x_k as an initial guess. It is known from [25, 27, 71] that if the preconditioned matrix $P^{-1}\hat{A}$ is diagonal then the kth residual $r^{(k)}$ of the preconditioned GMRES

$$\frac{\|r^{(k)}\|_{P^{-1}_{\alpha_1,\alpha_2}\hat{A}}}{\|r^{(0)}\|_{P^{-1}_{\alpha_1,\alpha_2}\hat{A}}} \leq \kappa(V)\min_{p_k\in\Pi_k, p_k(0)=1}\max_{\lambda_j\in\sigma(P^{-1}_{\alpha_1,\alpha_2}\hat{A})}|p_k(\lambda_j)|, \tag{5.29}$$

where $\kappa(V) = \|V\|\|V^{-1}\|$, $P^{-1}\hat{A} = V\Lambda V^{-1}$, Λ is a diagonal matrix of eigenvalues of $P^{-1}\hat{A}$ and V is the matrix whose columns are the eigenvectors and Π_k is the set of polynomials of degree less than or equal k.

In this work, we experiment with $k = 3$ (see section 5.9.4). That is, GMRES stops after 3 steps and restarts with x_3 as a new initial guess [36]. In GMRES(3), (5.29) gives

$$\begin{aligned}\frac{\|r^{(3)}\|_{P^{-1}_{\alpha_1,\alpha_2}\hat{A}}}{\|r^{(0)}\|_{P^{-1}_{\alpha_1,\alpha_2}\hat{A}}} &\leq \kappa(V)\min_{p_3\in\Pi_k, p(0)=1}\max_{\lambda_j\in\sigma(P^{-1}_{\alpha_1,\alpha_2}\hat{A})}|p_3(\lambda_j)|, \\ &\leq \kappa(V)\max_{\lambda_j\in\sigma(P^{-1}_{\alpha_1,\alpha_2}\hat{A})}|S_3(\lambda_j)|,\end{aligned} \tag{5.30}$$

for any cubic polynomial $S_3(x) \in \Pi_3$ satisfying $S_3(0) = 1$. We pick a specific cubic polynomial

$$q(x) = (x-1)(Ax^2 + (B-A)x + (C+B-A)),$$

which gives a good approximation to the min max value. Table 8.1 shows the min max value for different value of α_1, α_2 and h.

α_1, α_2, h	$\min\max$
$\alpha_1 = 2, \alpha_2 = 3, h = \frac{1}{8}$	0.0001587299
$\alpha_1 = 2, \alpha_2 = 3, h = \frac{1}{16}$	0.0001588353
$\alpha_1 = 4, \alpha_2 = 6, h = \frac{1}{32}$	0.5845693725

Table 5.1: The min max value for different values of α_1, α_2 and h.

The last column in this table shows how small the min max value is.

Remark: The preconditioning can be generalized for the stochastic case.

5.9 Numerical Computation

In this section, we compute the eigenvalues bounds for the matrix generated by finite element matrices of problem (5.1) to verify the results in Theorem 5.1. Also, we solve some problem using the techniques denoted in section 5.7 and section 5.8.

5.9.1 Computational Eigenvalues bounds

We study (5.1) on $\Omega = [0,1] \times [0,1]$ with different uniform meshes and different values of the parameters α_1 and α_2. Throughout we use permeabilities as test cases:

1. A^{-1}=A_c^{-1}= constant.

2. A^{-1}=A_r^{-1}= randomly generated.

3. $A^{-1} = A_t^{-1}$ where

$$A_t^{-1} = A^{-1}(x, y) = (1 + 2\epsilon \mathrm{Cos}(\pi x)\mathrm{Cos}(\pi y) + \epsilon^2 \mathrm{Cos}^2(\pi y)).$$

4. $A^{-1} = A_l^{-1}$ where

$$A_l^{-1} = \begin{cases} \epsilon & 0 < x < 0.5 - \delta \\ g(x) + \epsilon & 0.5 - \delta \leq x < 0.5 + \delta \\ 1 + \epsilon & 0.5 + \delta \leq x \leq 1 \end{cases},$$

where $g(x)$ is cubic function such that $A^{-1} \in C^1([0,1]^2)$. The permeability A_r^{-1} is an array of random value generated wing the MATLAB function rand. The permeability A_t^{-1} was used in [37]. A_t^{-1} ranges from $(1-\epsilon)^{-2}$ to $(1+\epsilon)^{-2}$ with a min to max variation of $\frac{4\epsilon}{(1-\epsilon^2)^2}$. With $\epsilon = 0.99$, this variation is of the order of 10^7. A_l^{-1} has localized variation of $1 + \frac{1}{\varepsilon}$ in the region of $[0.5-\delta, 0.5+\delta] \times [0, 1]$. With $\epsilon = 0.01$, A_l^{-1} has strong localized variation. In all cases, we set $f = 1$. Preconditioning is applied with a stopping tolerance of 10^{-6} in tams of the residual error in the norm $\| \cdot \|_{P_{\alpha_1,\alpha_2}}$. All computations were performed on MATLAB Version 10 on Windows 7; computer inter(R) Core (TM) is CPU 2.53 GHz with 4.00 GB RAM. When $A^{-1} = I$, we observe that the eigenvalues of the preconditioned matrix $\{\lambda_1, ..., \lambda_{n+m}\}$ are tabled in Table 5.2- 5.10,; they confirm that the bounds in Theorem 5.1 are tight. Table 5.1 shows the bounds by using varying mesh sizes at the corresponding values of α_1, α_2 by choosing $\epsilon = 0.01$ and $c = 1$ in the case $\alpha_1\alpha_2 = 1$.

h	α_1	α_2	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	2.4757	0.3998	$[-1.0097, -0.9899]$	$(-1.0100, -0.9900]$	1	1
$\frac{1}{16}$	2.4997	0.3960	$[-1.0100, -0.9902]$	$(-1.0102, -0.9902]$	1	1
$\frac{1}{32}$	2.5057	0.3951	$[-1.0098, -0.9901]$	$(-1.0101, -0.9901]$	1	1

Table 5.2: Eigenvalues of the preconditioned matrix; unit coefficients at the corresponding values of α_1, α_2.

The interval in column 4 is a subset of the corresponding interval in column 5. Table 5.3 shows the bounds by using varying mesh sizes at the corresponding values of α_1, α_2 by choosing $\epsilon = 0.01$ and $c = 1$ in the case $\alpha_1\alpha_2 \neq 1$.

h	α_1	α_2	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	157.988	3	$[-0.0021098, -0.0021092]$	$(-0.0021099, -0.0021092]$	1	1
$\frac{1}{16}$	8.676	10	$[-0.0115264, -0.0114599]$	$(-0.0115266, -0.0114599]$	1	1
$\frac{1}{32}$	1.923	20	$[-0.0260015, -0.02533514]$	$(-0.0260020, -0.02533514]$	1	1

Table 5.3: Eigenvalues of the preconditioned matrix; unit coefficients at the corresponding values of α_1, α_2.

The interval in column 4 is a subset of the corresponding interval in column 5. Table 5.4 shows the bounds by varying the mesh sizes with $\alpha_1 = 2,\ \alpha_2 = 1/2$ $(\alpha_1\alpha_2 = 1)$.

h	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	$[-0.9997, -0.9756]$	$(-1.0000, -0.9756]$	1	1
$\frac{1}{16}$	$[-0.9999, -0.9754]$	$(-1.0000, -0.9754]$	1	1
$\frac{1}{32}$	$[-0.9999, -0.9753]$	$(-1.0000, -0.9753]$	1	1

Table 5.4: Eigenvalues of the preconditioned matrix; unit coefficients with $\alpha_1 = 2,\ \alpha_2 = 1/2\ (\alpha_1\alpha_2 = 1)$.

The interval in column 2 is a subset of the corresponding interval in column 3. Table 5.5 shows the bounds for variable mesh size with $\alpha_1 = 2,\ \alpha_2 = 3\ (\alpha_1\alpha_2 \neq 1)$. Here $\frac{\lambda_{\max}}{\lambda_{\min}} \sim 6$.

h	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	$[-0.166612, -0.162600]$	$(-0.166666, -0.162600]$	1	1
$\frac{1}{16}$	$[-0.166653, -0.162562]$	$(-0.166666, -0.162562]$	1	1
$\frac{1}{32}$	$[-0.166663, -0.162552]$	$(-0.166666, -0.162552]$	1	1

Table 5.5: Eigenvalues of the preconditioned matrix; unit coefficients with $\alpha_1 = 2,\ \alpha_2 = 3\ (\alpha_1\alpha_2 \neq 1)$.

Table 5.6 lists the eigenvalues of the preconditioned matrix $P_{\alpha_1\alpha_2}^{-1}\hat{A}$ with $h = \frac{1}{32}$ and different values of the parameters $\alpha_1, \alpha_2\ (\alpha_1\alpha_2 \neq 1)$.

α_1	α_2	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
1	3	$[-0.333319, -0.317272]$	$(-0.333333, -0.317272]$	1	1
2	3	$[-0.166663, -0.162552]$	$(-0.166666, -0.162552]$	1	1
4	6	$[-0.0416662, -0.0411459]$	$(-0.0416666, -0.0411459]$	1	1

Table 5.6: Eigenvalues of the preconditioned matrix; unit coefficients with $h = \frac{1}{32}$ and different values of the parameters α_1, α_2 ($\alpha_1\alpha_2 \neq 1$).

In practical application, the permeability function is random. we consider a problem where the function $A^{-1}(x, y)$ is random, A^{-1}=A_r^{-1}. We use the Matlab command rand to generate a random permeability. Table 5.7 shows the bounds by using varying mesh sizes and the corresponding values of α_1, α_2 by choosing $\epsilon = 0.01$, $c = 1$.

h	α_1	α_2	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	2.4757	0.3998	$[-1.0102, -0.9952]$	$(-1.0103, -0.9952]$	1	1
$\frac{1}{16}$	2.4997	0.3960	$[-1.0101, -0.9954]$	$(-1.0102, -0.9954]$	1	1
$\frac{1}{32}$	2.5057	0.3951	$[-1.01008, -0.9951]$	$(-1.01009, -0.9951]$	1	1

Table 5.7: Eigenvalues of the preconditioned matrix; random coefficients at the corresponding values of α_1, α_2.

The interval in column 4 is a subset of the corresponding interval in column 5. Table 5.8 shows the bounds by varying the mesh sizes with $\alpha_1 = 2,\ \alpha_2 = 1/2$ ($\alpha_1\alpha_2 = 1$).

h	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	$[-0.9997, -0.9819]$	$(-1.0000, -0.9819]$	1	1
$\frac{1}{16}$	$[-0.9999, -0.9817]$	$(-1.0000, -0.9817]$	1	1
$\frac{1}{32}$	$[-0.9999, -0.9815]$	$(-1.0000, -0.9815]$	1	1

Table 5.8: Eigenvalues of the preconditioned matrix; random coefficients with $\alpha_1 = 2,\ \alpha_2 = 1/2\ (\alpha_1\alpha_2 = 1)$.

The interval in column 2 is a subset of the corresponding interval in column 3. Table 5.9 shows the bounds by varying the mesh sizes with $\alpha_1 = 2,\ \alpha_2 = 3$ $(\alpha_1\alpha_2 \neq 1)$.

h	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{\min}})]$	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	$[-0.1666294, -0.1636461]$	$(-0.1666666, -0.1636461]$	1	1
$\frac{1}{16}$	$[-0.1666575, -0.1635818]$	$(-0.1666666, -0.1635818]$	1	1
$\frac{1}{32}$	$[-0.1666645, -0.1635992]$	$(-0.1666666, -0.1635992]$	1	1

Table 5.9: Eigenvalues of the preconditioned matrix; random coefficients with $\alpha_1 = 2,\ \alpha_2 = 3\ (\alpha_1\alpha_2 \neq 1)$.

Table 5.10 lists the eigenvalues of the preconditioned matrix $P_{\alpha_1\alpha_2}^{-1}\hat{A}$ with $h = \frac{1}{32}$ and different values of the parameters α_1, α_2 $(\alpha_1\alpha_2 \neq 1)$.

α_1	α_2	$[\lambda_1, \lambda_m]$	$(-\frac{1}{\alpha_1\alpha_2}, -\frac{1}{\alpha_2}(\frac{\mu_{\min}}{\lvert K\rvert_{\min}+\alpha_1\mu_{min}})]$	λ_{m+1}	λ_{m+n}
1	3	$[-0.3333243, -0.32119389]$	$(-0.3333333, -0.32119389]$	1	1
2	3	$[-0.1666645, -0.1635992]$	$(-0.1666666, -0.1635992]$	1	1
4	6	$[-0.0416664, -0.0412779]$	$(-0.0416666, -0.0412779]$	1	1

Table 5.10: Eigenvalues of the preconditioned matrix; random coefficients with $h = \frac{1}{32}$ and different values of the parameters α_1, α_2 ($\alpha_1\alpha_2 \neq 1$).

5.9.2 Preconditioned MINRES

Here, we solve problem (5.1) with homogeneous Dirichlet boundary conditions. The domain of the problem is the unit square $\Omega = [0,1] \times [0,1]$. We use the Ravirt-Thomas on rectangular meshes. We solve the problem with different types of permeability $A_c^{-1}, A_r^{-1}, A_t^{-1}, A_l^{-1}$ and different choices of parameter α_1, α_2. Iteration counts of the preconditioned MINRES are listed in Table 5.11.

A^{-1}	P\h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
A_C^{-1}	$\alpha_1 = 1, \alpha_2 = 1$	4	4	4	4
	$\alpha_1 = 3, \alpha_2 = \frac{1}{\alpha_1}$	3	3	3	3
	$\alpha_1 = 2.4757, \alpha_2 = 0.4039$	3	3	3	3
	I	68	163	331	634
A_r^{-1}	$\alpha_1 = 1, \alpha_2 = 1$	5	5	5	5
	$\alpha_1 = 3, \alpha_2 = \frac{1}{\alpha_1}$	4	4	4	4
	$\alpha_1 = 2.4757, \alpha_2 = 0.4039$	4	4	4	4
	I	215	419	674	1267
A_t^{-1}	$\alpha_1 = 1, \alpha_2 = 1$	4	4	4	4
	$\alpha_1 = 3, \alpha_2 = \frac{1}{\alpha_1}$	3	3	3	3
	$\alpha_1 = 2.4757, \alpha_2 = 0.4039$	3	3	3	3
	I	66	161	325	622
A_l^{-1}	$\alpha_1 = 1, \alpha_2 = 1$	4	4	4	4
	$\alpha_1 = 3, \alpha_2 = \frac{1}{\alpha_1}$	3	3	3	3
	$\alpha_1 = 2.4757, \alpha_2 = 0.4039$	3	3	3	3
	I	68	163	331	634

Table 5.11: MINRES iterations; homogeneous Dirichlet condition with different types of permeability $A_c^{-1}, A_r^{-1}, A_t^{-1}, A_l^{-1}$ and different choices of parameter α_1, α_2.

We observe that varying the permeability types and varying the mesh size does not affect the number of iterations for the preconditioned problems. The smallest number of iterations occur when $(\alpha_1, \alpha_2) = (2.4757, 4039)$ or $(3, \frac{1}{3})$.

5.9.3 Preconditioned GMRES

Here, we solve problem (5.1) with homogeneous Dirichlet boundary conditions. The domain of the problem is the unit square $\Omega = [0,1] \times [0,1]$. We use the Ravirt-Thomas on rectangular meshes. We solve the problem with different types of permeability $A_c^{-1}, A_r^{-1}, A_t^{-1}, A_l^{-1}$ and different choices of parameter α_1, α_2. Iteration counts of the preconditioned GMRES are listed in Table 5.12.

A^{-1}	P\h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
A_C^{-1}	$\alpha_1 = 1, \alpha_2 = \frac{1}{3}$	4	4	4	4
	$\alpha_1 = 4, \alpha_2 = 6$	3	3	3	3
	$\alpha_1 = 157.988, \alpha_2 = 3$	2	2	2	2
	I	24	76	165	310
A_r^{-1}	$\alpha_1 = 1, \alpha_2 = \frac{1}{3}$	4	4	4	4
	$\alpha_1 = 4, \alpha_2 = 6$	3	3	3	3
	$\alpha_1 = 157.988, \alpha_2 = 3$	2	2	2	2
	I	114	221	439	843
A_t^{-1}	$\alpha_1 = 1, \alpha_2 = \frac{1}{3}$	4	4	4	4
	$\alpha_1 = 4, \alpha_2 = 6$	3	3	3	3
	$\alpha_1 = 157.988, \alpha_2 = 3$	2	2	2	2
	I	24	75	163	308
A_l^{-1}	$\alpha_1 = 1, \alpha_2 = \frac{1}{3}$	4	4	4	4
	$\alpha_1 = 4, \alpha_2 = 6$	3	3	3	3
	$\alpha_1 = 157.988, \alpha_2 = 3$	2	2	2	2
	I	24	76	164	310

Table 5.12: GMRES iterations; homogeneous Dirichlet condition with different types of permeability $A_c^{-1}, A_r^{-1}, A_t^{-1}, A_l^{-1}$ and different choices of parameter α_1, α_2.

We note that 2-4 iterations are needed for required cases for all different types of permeability and different mesh size.

5.9.4 Preconditioned GMRES(3)

In this subsection we experiment with GMRES(3). Consider the case $A^{-1} = A_t^{-1}$. Table 5.13 shows the preconditioned residual norm of every iterations.

outer iter.	inner iter.	$\|P_{\alpha_1,\alpha_2}^{-1} r^{(i,j)}\|, h = \frac{1}{32}$
0	0	$1.0923e + 004$
1	1	$1.2943e + 003$
1	2	$2.4122e + 001$
1	3	$7.0452e - 002$
2	1	$1.8581e - 002$
2	2	$2.7800e - 004$
2	3	$2.0807e - 006$
3	1	$2.0698e - 006$
3	2	$5.2031e - 008$
3	3	$1.7957e - 010$

Table 5.13: GMRES(3): Preconditioned residual norm for A_t^{-1}.

Figure 5.2 shows the logarithm of the norm of the preconditioned residual .vs. iteration counts for various choices of the parameters α_1 and α_2. The green line corresponding to $(\alpha_1, \alpha_2) = (3, 1/3)$, the red line corresponding to $(\alpha_1, \alpha_2) = (2, 3)$ and the blue line corresponding to $(\alpha_1, \alpha_2) = (4, 6)$.

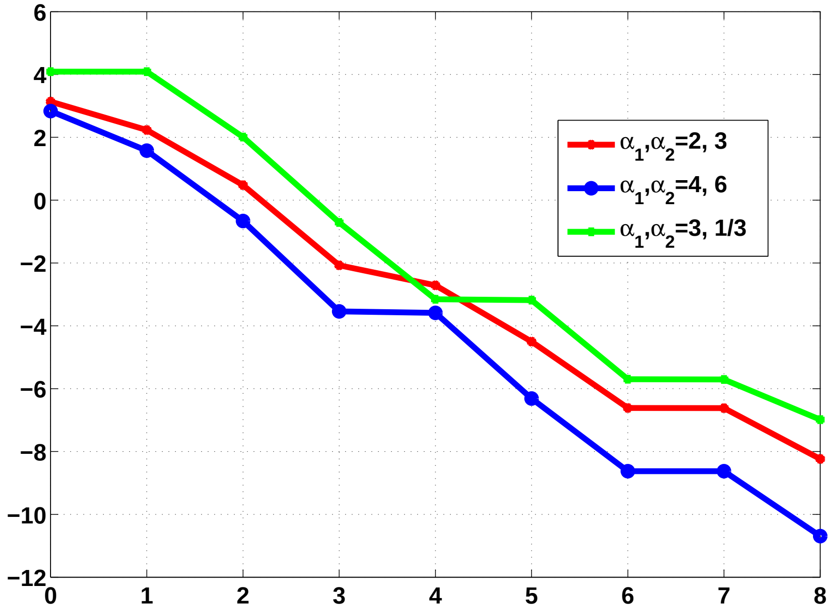

Figure 5.2: $log(\| P^{-1}_{\alpha_1,\alpha_2} r \|)$ .vs. Iteration counts.

The last case we consider a permeability function $A^{-1} = A_l^{-1}$. Table 5.14 shows the preconditioned residual norm of every iterations.

outer iter.	inner iter.	$\lVert P_{\alpha_1,\alpha_2}^{-1} r^{(i,j)} \rVert, h = \frac{1}{32}$
0	0	$1.0923e+004$
1	1	$1.2937e+003$
1	2	$2.0361e+001$
1	3	$5.0267e-002$
2	1	$1.2283e-002$
2	2	$1.6651e-004$
2	3	$1.0621e-006$
3	1	$1.0563e-006$
3	2	$2.2258e-008$
3	3	$6.4786e-011$

Table 5.14: GMRES(3): Preconditioned residual norm for A_l^{-1}.

Figure 5.3 shows the logarithm of the norm of the preconditioned residual .vs. iteration numbers for different values of the parameters α_1 and α_2. The green line corresponding to $(\alpha_1, \alpha_2) = (3, 1/3)$, the red line corresponding to $(\alpha_1, \alpha_2) = (2, 3)$ and the blue line corresponding to $(\alpha_1, \alpha_2) = (4, 6)$.

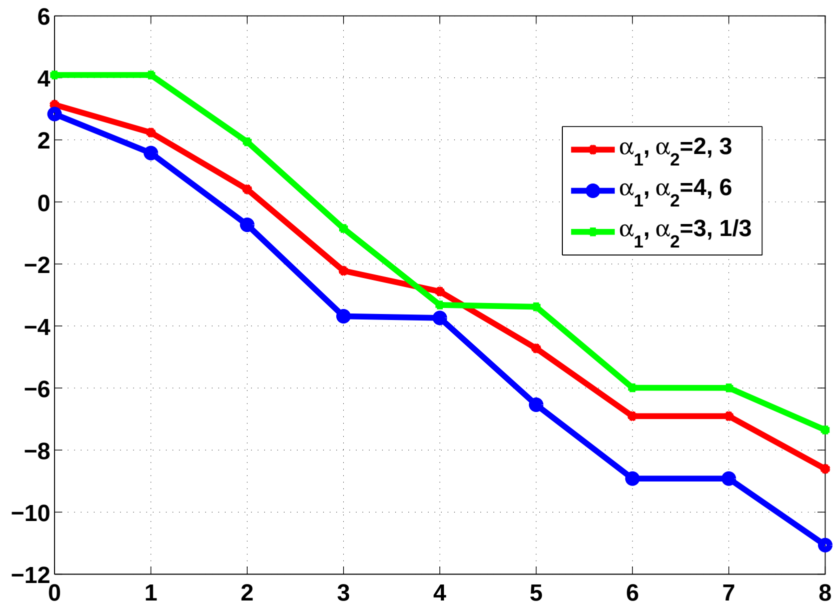

Figure 5.3: $log(\| P_{\alpha_1,\alpha_2}^{-1} r \|)$ .vs. Iteration counts.

CHAPTER 6

NUMERICAL EXPERIMENTS

We solve the stochastic Darcy Equation (1.1) with stochastic right-hand side by using the stochastic Galerkin Method discussed in Chapter 2. We begin this chapter by constructing the Galerkin approximation for the problem (1.1) in Section 6.1. This is followed by, building the Linear system in Section 6.2. Finally, in Section 6.3 we illustrate the numerical results with real permeability data from a sandstone core.

6.1 Linear system.

In this Section, we study the structure of coefficient matrix associated with (4.38). The restriction of the stochastic variability to finite dimensional basis and taking the inner product with respect to the random part reduces the stochastic saddle point problem (4.9) to a deterministic one without the 2^M dimensional parameter ω. The Galerkin discretization is obtained by restricting the test functions in (4.9) to finite dimensional subspaces of the deterministic tensor product space $V \times W$ which are subsets of $H_0(\mathrm{div}; D)$, $L^2(D)$ and $L^2(\Omega, \mathcal{F}, P)$. Choosing subspaces require the discrete version satisfy (4.27). $L^2(\Omega, \mathcal{F}, P)$ can be discretized independently. More precisely, let the basis for $X_{h\nu}^{\text{div}}$, $X_{h\nu}^{0}$ and S_M denoted by

$$\begin{aligned}
\Phi_h &= \text{span}\,\{\varphi_i : i = 1, ..., N_u\} \subset H_0(div, D), \\
\phi_h &= \text{span}\,\{\phi_i : i = 1, ..., N_p\} \subset L^2(D), \\
S_M &= \text{span}\left\{Z_k :\ 1 \le k \le 2^M\right\} \subset L^2(\Omega, \mathcal{F}, P),
\end{aligned} \tag{6.1}$$

where the subscripts h refer to the discretization parameters, and define

$$\begin{aligned}
V_{h\nu,M} &= \left\{r(x,\omega) \in \text{span}\{\varphi_i(x)Z_k(\omega) : i = 1, ..., N_u,\ 1 \le k \le 2^M\}\right\} = \Phi_h \otimes S_M, \\
W_{h\nu,M} &= \left\{v(x,\omega) \in \text{span}\{\phi_l(x)Z_k(\omega) : l = 1, ..., N_p,\ 1 \le k \le 2^M\}\right\} = \phi_h \otimes S_M,
\end{aligned} \tag{6.2}$$

where $\dim(V_{h\nu,M} \times W_{h\nu,M}) = N_\omega(N_u + N_p)$, $N_\omega = 2^M$. For the subspaces Φ_h and ϕ_h, we use the zero-order Raviart-Thomas mixed approximation. Refer to Chapter 5 and the discussion in [52]. Based on the partition τ_h of the space domain D into element of rectangular shape with maximal diameter $h > 0$. More explicitly, given a partition τ_h of D into rectangular subdomains, the corresponding to the Raviart-Thomas element have the form $\varphi_i = \binom{c_1+c_2x}{c_3+c_4y}$. Then

$$\Phi_h := \{u \in H_0(\text{div}; D) :\ u|_K \in Q_{1,0}(K) \times Q_{0,1}(K)\ \forall K \in \tau_h\},$$

where $Q_{i,j}$ denotes the space of polynomials of degree i in the first spatial variable and j is the second spatial variable. In this case we construct a vector that is piecewise linear in each component. Moreover, the normal component is continuous, across the edge of the element of τ_h. Similarly, the basis $\{\phi_i\}_{i=1}^{N_p}$ are piecewise constants. Then

$$\phi_h := \{p \in L^2(D) : p|_K = c, \quad \forall K \in \tau_h\},$$

6.1.1 Computation of Stochastic Integrals.

Suppose $h(\omega)$ is a random variable in $L^2(\Omega, \mathcal{F}, P)$, its expected value is numerically approximated by

$$\int_\Omega h(\omega) dP(\omega) = \frac{1}{N} \sum_{i=1}^{N} h(\omega_i) \tag{6.3}$$

in accordance with the Low of Large numbers. This approximation is also Known as Monte Carlo integration.

6.1.2 Saddle Point Problem.

We start by expressing $u_{h,\nu}^{(M)}$, $p_{h,\nu}^{(M)}$ in terms of basis as,

$$\begin{aligned} u_{h,\nu}^{(M)} &= \sum_{i=1}^{N_u} \sum_{j=1}^{2^M} u_{i,j} \varphi_i(x) Z_j(\omega), \\ p_{h,\nu}^{(M)} &= \sum_{i=1}^{N_p} \sum_{j=1}^{2^M} p_{i,j} \phi_i(x) Z_j(\omega). \end{aligned} \tag{6.4}$$

Inserting (4.12) and (6.4) into (4.38) yelids

$$\sum_{i=1}^{N_u}\sum_{j=1}^{2^M} u_{i,j}\left\langle (A^{-1}(x,\omega)\varphi_i(x)Z_j(\omega), \varphi_{\acute{i}}(x)Z_{\acute{j}}(\omega))\right\rangle +$$
$$\sum_{i=1}^{N_p}\sum_{j=1}^{2^M} p_{i,j}\left\langle ((\varphi_{\acute{i}}(x)Z_{\acute{j}}(\omega), \nabla\cdot\phi_i(x)Z_j(\omega))\right\rangle = 0, \tag{6.5}$$
$$\sum_{i=1}^{N_u}\sum_{j=1}^{2^M} u_{i,j}\left\langle (\nabla\cdot\varphi_i(x)Z_j(\omega), \phi_{\acute{k}}(x)Z_{\acute{j}}(\omega))\right\rangle = -\langle (f, \phi_{\acute{k}}(x)Z_{\acute{j}})\rangle,$$

where, $\acute{i} = 1, ..., N_u$, $\acute{k} = 1, ..., N_p$ and $\acute{j} = 1, ..., N_\omega$. The result can be expressed in the matrix saddle point problem

$$\begin{bmatrix} \hat{A} & \hat{B}^T \\ \hat{B} & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}. \tag{6.6}$$

In what follow we describe the various components of (6.6). The solution vector consists of two block vectors

$$\begin{bmatrix} [u_1] \\ [u_2] \\ . \\ . \\ . \\ [u_{N_\omega}] \end{bmatrix} \in \mathbb{R}^{N_u N_\omega} \quad \text{and} \quad \begin{bmatrix} [p_1] \\ [p_2] \\ . \\ . \\ . \\ [p_{N_\omega}] \end{bmatrix} \in \mathbb{R}^{N_p N_\omega}.$$

We have

$$
\begin{aligned}
&[u_j]_i = u_{ij}, \quad i = 1, 2, ..., N_u, \quad 1 \leq j \leq 2^M, \\
&[p_j]_i = p_{ij}, \quad i = 1, 2, ..., N_p, \quad 1 \leq j \leq 2^M.
\end{aligned} \tag{6.7}
$$

The matrices $\hat{A}$ and $\hat{B}$ in (6.6) can be written as

$$
\begin{aligned}
[\hat{A}]_{(\acute{i},\acute{j}),(i,j)} &= \left\langle (A^{-1}\varphi_i Z_j(\omega), \varphi_{\acute{i}} Z_{\acute{j}}(\omega)) \right\rangle \\
&= \left\langle (A^{-1}\varphi_i, \varphi_{\acute{i}}) Z_j(\omega) Z_{\acute{j}}(\omega) \right\rangle, \\
&= \left\langle \left(\left(E[A^{-1}] + \sum_{k=1}^{2^M} a_k Z_k(\omega) \right) \varphi_i, \varphi_{\acute{i}} \right) Z_j(\omega) Z_{\acute{j}}(\omega) \right\rangle, \\
&= E[A^{-1}](\varphi_i, \varphi_{\acute{i}}) \langle Z_j(\omega) Z_{\acute{j}}(\omega) \rangle + \sum_{k=1}^{2^M} (a_k(x)\varphi_i, \varphi_{\acute{i}}) \langle Z_k(\omega) Z_j(\omega) Z_{\acute{j}}(\omega) \rangle.
\end{aligned}
$$

Thus,

$$
[\hat{A}]_{(\acute{i},\acute{j}),(i,j)} = E[A^{-1}](\varphi_i, \varphi_{\acute{i}}) \langle Z_j(\omega) Z_{\acute{j}}(\omega) \rangle + \sum_{k=1}^{2^M} (a_k(x)\varphi_i, \varphi_{\acute{i}}) \langle Z_k(\omega) Z_j(\omega) Z_{\acute{j}}(\omega) \rangle, \tag{6.8}
$$

where, $i, \acute{i} = 1, ..., N_u$ and $j, \acute{j} = 1, ..., N_\omega$ and the coefficient $a_k(x)$ is computed by using the computation of stochastic integrals as

$$
\begin{aligned}
a_k(x) &= \int_\Omega A^{-1}(x, \omega) Z_k(\omega) dP(\omega), \\
&\cong \frac{1}{N} \sum_{l=1}^{N} A^{-1}(x, \omega_l) Z_k(\omega_l).
\end{aligned} \tag{6.9}
$$

Similarly,

$$
[\hat{B}]_{(\acute{i},\acute{j}),(i,j)} = -(\nabla \cdot \varphi_i, \varphi_{\acute{i}}) \langle Z_j(\omega) Z_{\acute{j}}(\omega) \rangle, \tag{6.10}
$$

where, $i = 1, ..., N_u$, $\acute{i} = 1, ..., N_p$ and $j, \acute{j} = 1, ..., N_\omega$. The integrals with respect to the independent variables ω and x can be separated. This property leads to a bilinear structure in (6.8)-(6.10). This leads to write the matrices $\hat{A}, \hat{B}$ as sums of Kronecker products

$$\hat{A} = G_0 \otimes A_0 + \sum_{k=1}^{2^M} G_k \otimes A_k, \qquad \hat{B} = G_0 \otimes B, \tag{6.11}$$

the factors of which are given by

$$\begin{aligned}
[A_0]_{i,j} &= (E[A^{-1}]\varphi_j, \varphi_i), && A_0 \in \mathbb{R}^{N_u \times N_u}, \quad i, j = 1, ..., N_u, \\
[A_k]_{i,j} &= (a_k(x)\varphi_j, \varphi_i), && A_k \in \mathbb{R}^{N_u \times N_u}, \quad i, j = 1, ..., N_u, \ k = 1, ..., 2^M, \\
[B]_{i,j} &= -(\nabla \cdot \varphi_i, \phi_j), && B \in \mathbb{R}^{N_p \times N_u}, \quad i = 1, ..., N_p, \quad j = 1, ..., N_u, \\
[G_0]_{\acute{j},j} &= \langle Z_j Z_{\acute{j}} \rangle, && G_0 \in \mathbb{R}^{N_\omega \times N_\omega} \ \ j, \acute{j} = 1, ..., N_\omega, \\
[G_k]_{\acute{j},j} &= \langle Z_k Z_j Z_{\acute{j}} \rangle, && j, \acute{j} = 1, ..., N_\omega, \ k = 1, ..., 2^M.
\end{aligned}$$

Note that the matrix A_0 constitute the (1,1)-blocks of the associated deterministic problem with $A^{-1} = E[A^{-1}]$. Since the input random field does not occur in the bilinear form $b(\cdot, \cdot)$, the matrix B is the (2,1)-block of the deterministic problem. The structure of the matrices G_0 and G_k depend on our basis Z_k. As we have seen in Chapter 4,

$$\langle Z_j Z_{\acute{j}} \rangle = \int_\Omega Z_j(\omega) Z_{\acute{j}}(\omega) dP(\omega) = \begin{cases} e - 1, & j = \acute{j}, \\ 0, & j \neq \acute{j}, \end{cases},$$

and

$$\langle Z_k Z_j Z_{\acute{j}} \rangle = \int_\Omega Z_k(\omega) Z_j(\omega) Z_{\acute{j}}(\omega) dP(\omega) = \begin{cases} e^3 - 1, & k = j = \acute{j}, \\ e - 1, & k = j \neq \acute{j} \text{ or } k \neq j = \acute{j} \text{ or } k = \acute{j} \neq j, \\ 0, & k \neq j \neq \acute{j}. \end{cases}$$

An analogous representation hold for f on the right-hand side of (6.6).

6.1.3 Computation of the right hand side.

We use the computation of stochastic integrals to compute f as,

$$\begin{aligned}
[f_{\acute{j}}]_{\acute{k}} &= -\langle (f, \phi_{\acute{k}}(x) Z_{\acute{j}}(\omega)) \rangle, \\
&= -\langle (f(x,\omega), \phi_{\acute{k}}(x) Z_{\acute{j}}(\omega)) \rangle, \\
&= -\int_D \phi_{\acute{k}}(x) \left\langle f(x,\omega) Z_{\acute{j}}(\omega) \right\rangle dx, \\
&= -\int_D \phi_{\acute{k}}(x) \left[\int_\Omega f(x,\omega) Z_{\acute{j}}(\omega) dP(\omega) \right] dx, \\
&\cong -\int_D \phi_{\acute{k}}(x) \left[\frac{1}{N} \sum_{l=1}^{N} f(x,\omega_l) Z_{\acute{j}}(\omega_l) \right] dx, \\
&\cong -\int_D \phi_{\acute{k}}(x) \left[\frac{1}{N} \sum_{l=1}^{N} f(x,\omega_l) Z_{\acute{j}}(\omega_l) \right] dx,
\end{aligned}$$

where, $\acute{k} = 1, ..., N_p$ and N is the number of generating.

6.2 Numerical results.

We illustrate our sGFEM with a stochastic problem in two dimensions with real permeability data. The problem is discretized in both spatial and random dimensions. The discretization in the spatial dimension is done in the same way as in the deterministic FEM as explained in section 6.1. In addition we build the SGFEM by adjoining the discretization of the random field over the standard discretization of the spatial dimension.

It is very difficult to find published data for the spatial distribution of permeability inside geologic media, such as rocks, because permeability cannot be measured nondestructively at a very fine (pore- or sub-pore) scale. In the literature, the only permeability information generally available is the measured average core permeability. To determine the detailed fine permeability distribution inside of a rock core, it must be indirectly calculated using another measured dataset, such as local porosity distribution (obtained from X-ray Computerized Tomography scan), or as in the new technique of Krause et al. [47] from pore throat size distribution (obtainable from Mercury Injection).

Krause et al. (2009) used a relatively homogeneous Berea sandstone core of 18.5% average porosity, and $A = 85$ md (millidarcy) permeability. The core was 8 inches long and two inches in diameter. They measured capillary pressure using mercury intrusion up to 30,000 psi pressure, converted it to capillary pressure for brine, and from the brine saturation versus capillary pressure curve they calculated the local value of permeability for every grid point inside the core. The grid size used by these authors was 1.27mm $\times$ 1.27mm $\times$ 3mm. As the pores contained a mixture of brine (salt water) and CO2 gas, by the classical Leverett [41] equation capillary pressure and saturation are related as

$$\mathbf{P}_c = \sigma_{\frac{CO2}{brine}} cos\theta_{\frac{CO2}{brine}} \sqrt{\frac{\Phi}{A}} J(S_\omega), \tag{6.12}$$

where

$$S_\omega = \frac{S_{brine} - S_{lr}}{1 - S_{lr}}, \tag{6.13}$$

$$J(S_\omega) = \mathbf{A}\left(\frac{1}{S_*^\lambda} - 1\right) + \mathbf{B}(1 - S_*^\mu). \tag{6.14}$$

In (6.12), $\sigma_{\frac{CO2}{brine}} = 28.5\,\text{dynes/cm}$ is interfacial tension between brine and CO2, $\theta_{\frac{CO2}{brine}}$ is the contact angle between brine and $CO2$, Φ is porosity, A permeability, S (with different subscripts) is saturation in all equations. In (6.13), S_{lr} is the residual liquid phase saturation. In (6.14), S^* is the normalized brine saturation, $\mathbf{A}, \mathbf{B}, \lambda, \mu$ are fitting parameters. From (6.12) the permeability can be calculated. In the study of Krause et al. [47] four calculations were made with different fitting parameters of the Leverett function, to estimate the permeability distribution within the core. The four permeability distributions are similar in their high permeability parts, but the lower permeability values were different (in an apparently random manner), so that we are justified to consider the four distributions as real-world examples for different realizations of a random permeability field. Figure 6.1 shows the four realizations of the permeability distributions in the Krause et. al. paper. Colors represent the local permeability values in md (millidarcy) unit.

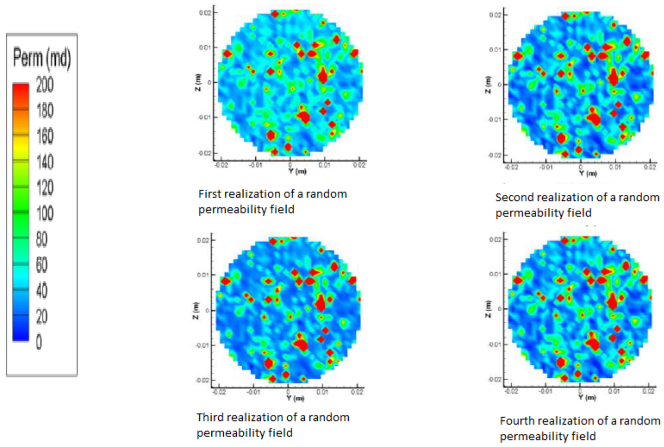

Figure 6.1: Four realizations of the random permeability field. (From Krause et. al.).

The permeability distributions in figure 6.1 are dominated primarily by small regions of very high permeability (in red) and very low permeability regions (in blue) with intermediate permeability in between.

We study (1.1) on $D = [0,1] \times [0,1]$ for the four different realizations of the spatial distribution of the permeability field inside the rock sample shown in figure 6.1. We compute the permeability distribution and the pressure distribution for the mean value of the random permeability field. Also we evaluate the pressure distributions and the velocity distribution for the four realizations of the random permeability field generated by randomly modifying. The random modification was made by adding to the original permeability values a suitable normalized pseudo random normally distributed, zero mean valued, quantity. The results will be described in the examples that follow. All computations were performed on MATLAB Version 10 on Windows 7; computer inter(R) Core (TM) is CPU

2.53 GHz with 4.00 GB RAM.

Example 6.1: Using the first realization of the random permeability field in Figure 6.1, we obtain the results described in the following figures. Figure 6.2 shows the first realization of the permeability distribution.

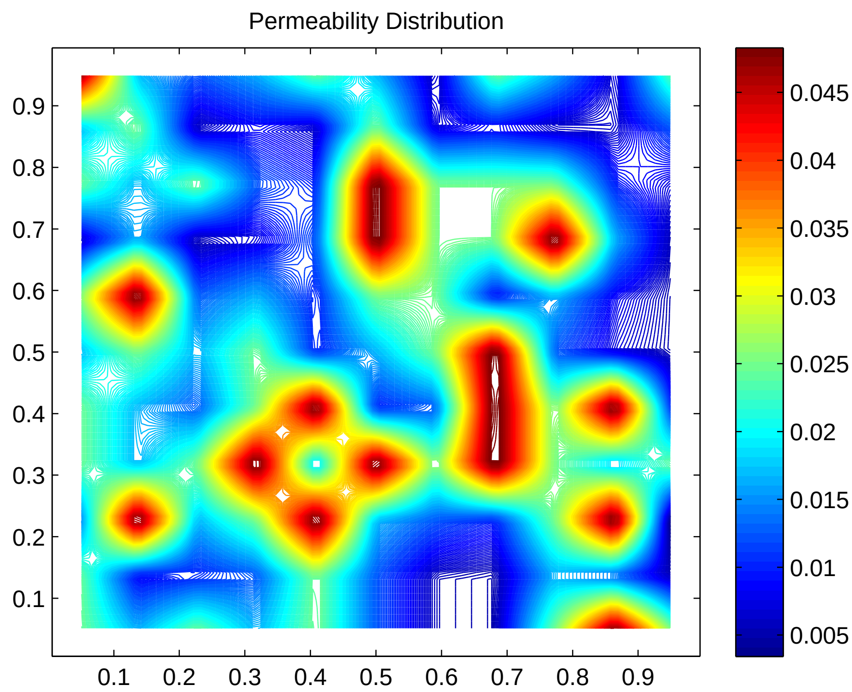

Figure 6.2: The first realization of the permeability distribution.

Figure 6.3 show the pressure distributions corresponding to this realization of the permeability field.

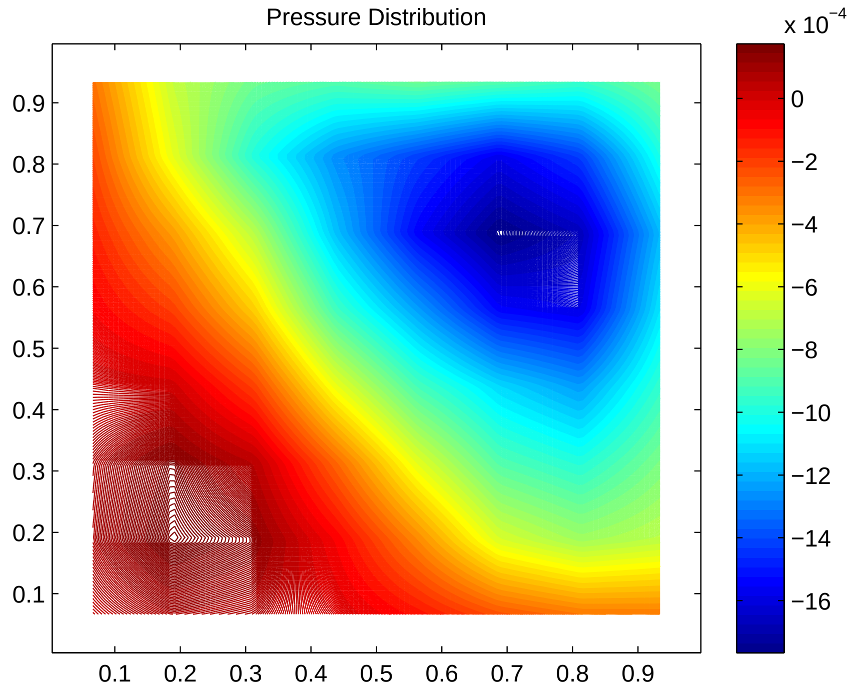

Figure 6.3: The pressure distributions for the first realization of the permeability field.

Figure 6.4 shows the pressure distribution corresponding to randomly modified permeability field from the firs realization.

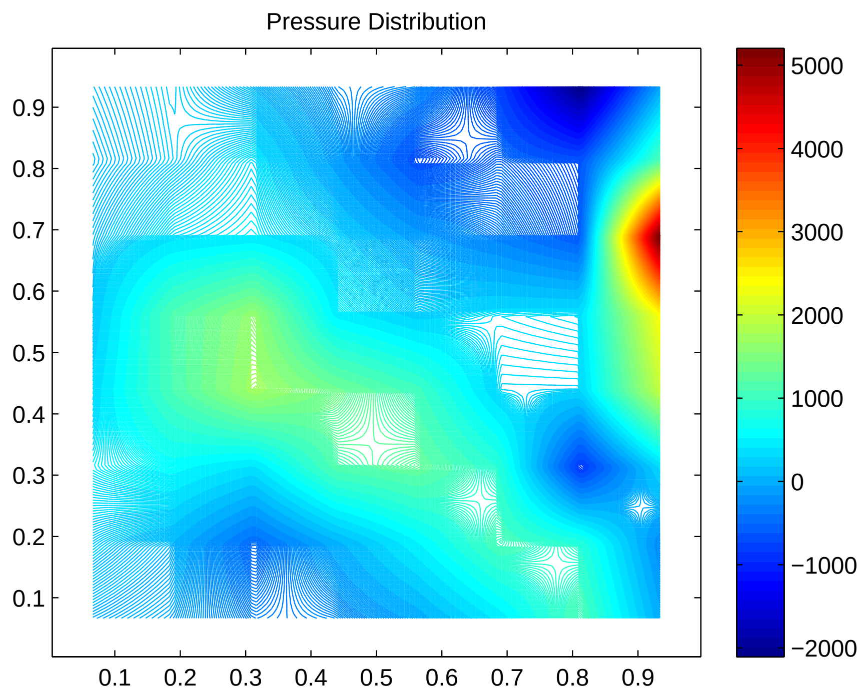

Figure 6.4: The pressure distribution corresponding to the randomly modified first realization of the permeability field.

Figure 6.5 shows that the pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generated random permeability field of the first realization.

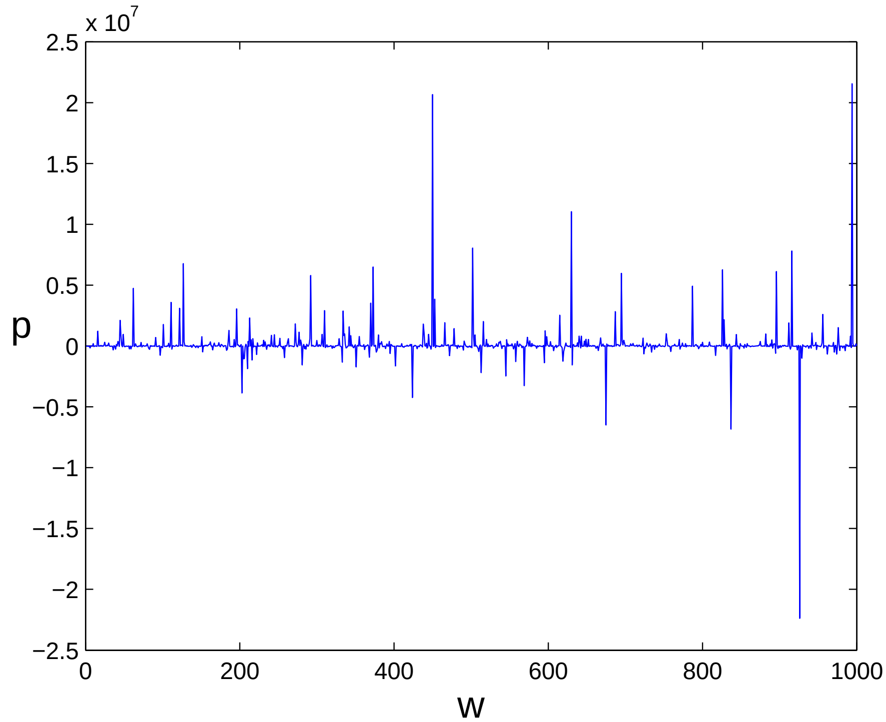

Figure 6.5: The pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generating random permeability field of the first realization.

Figure 6.6 shows the velocity distribution corresponding to the randomly modified first realization of the permeability field.

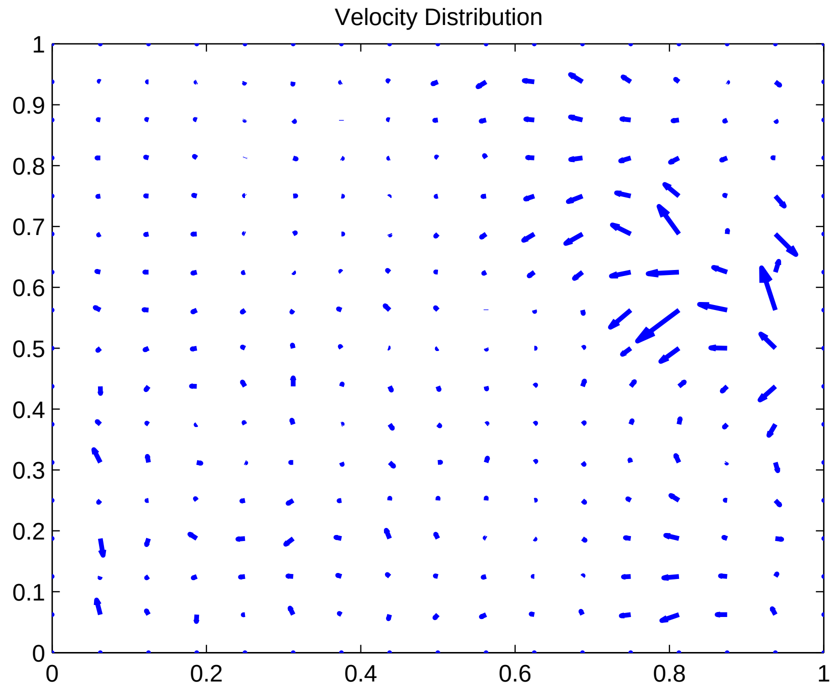

Figure 6.6: The velocity distribution for the randomly modified first realization of the permeability field.

Example 6.2: Using the second realization of the random permeability field in figure 6.1, we obtain the following results described in the next figures. Figure 6.7 shows the second realization of the permeability distribution.

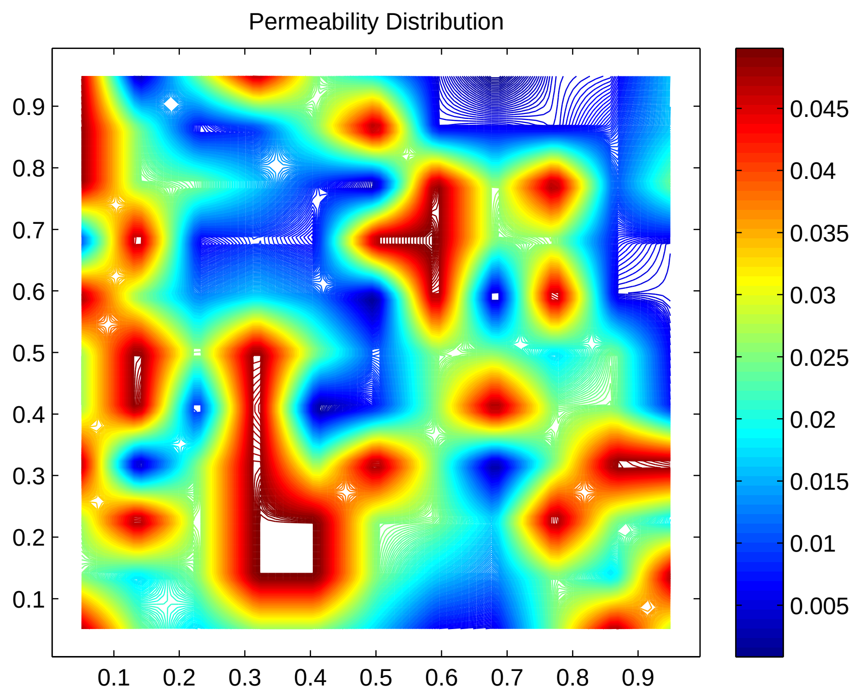

Figure 6.7: The second realization of the permeability distribution.

Figure 6.8 shows the pressure distributions corresponding to this realization of the permeability field.

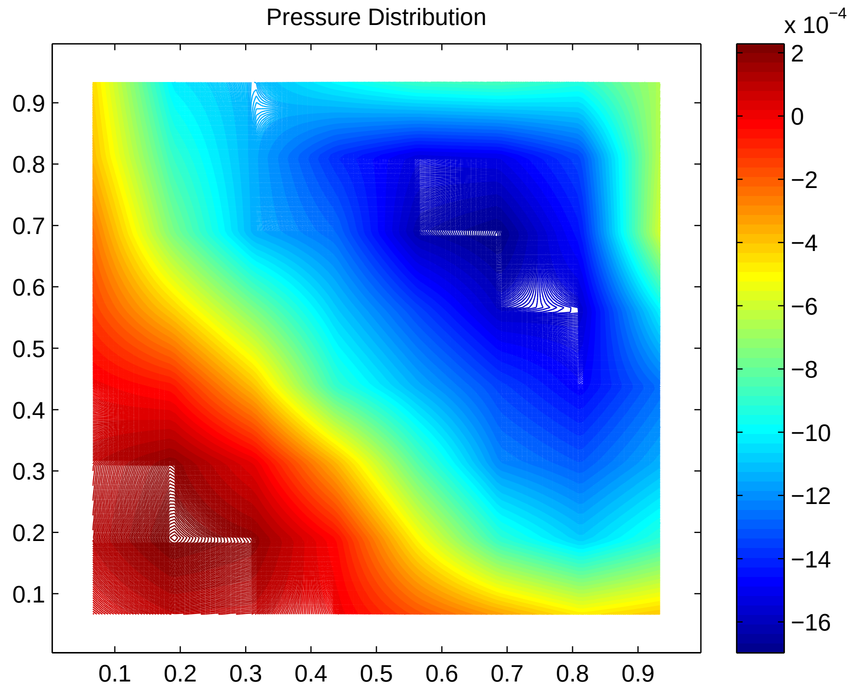

Figure 6.8: The pressure distributions for the second realization of the permeability field.

Figure 6.9 shows the pressure distribution corresponding to randomly modified permeability field from the second realization.

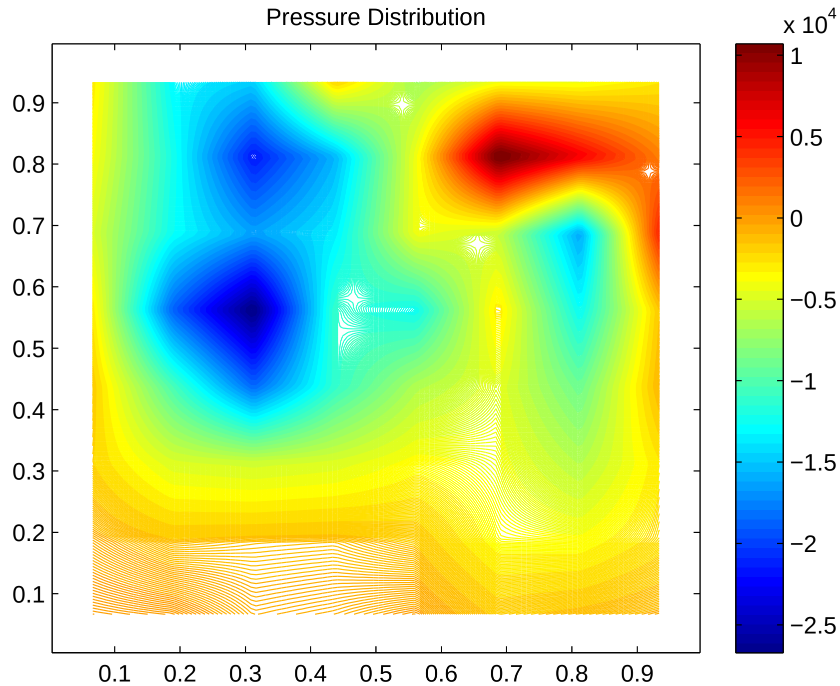

Figure 6.9: The pressure distribution of the corresponding to the randomly modified second realization of the permeability field.

Figure 6.10 shows that the pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generated random permeability field of the second realization.

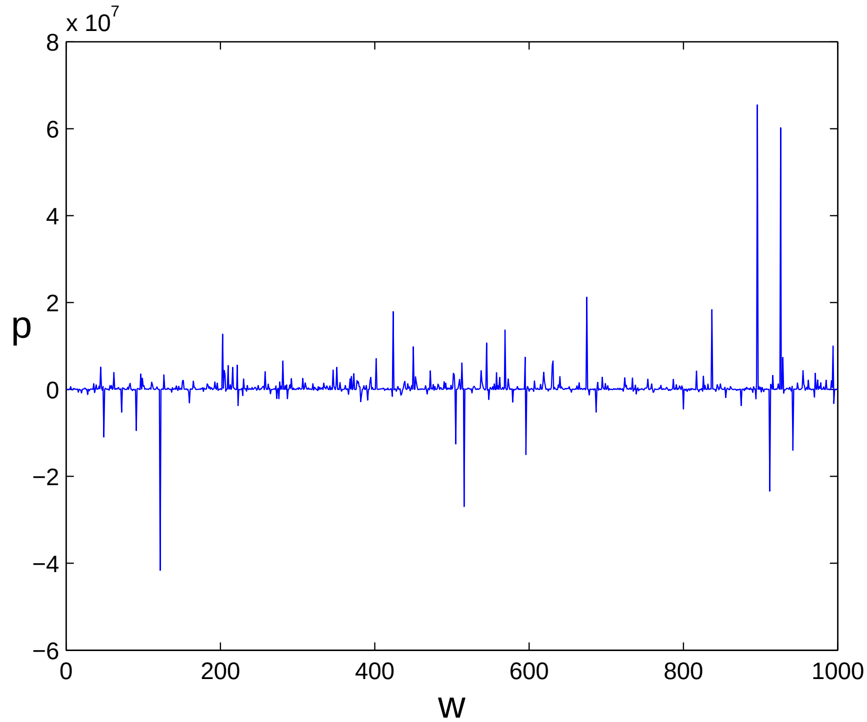

Figure 6.10: The pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generating random permeability field of the second realization.

Figure 6.11 shows the velocity distribution corresponding to the randomly modified second realization of the permeability field.

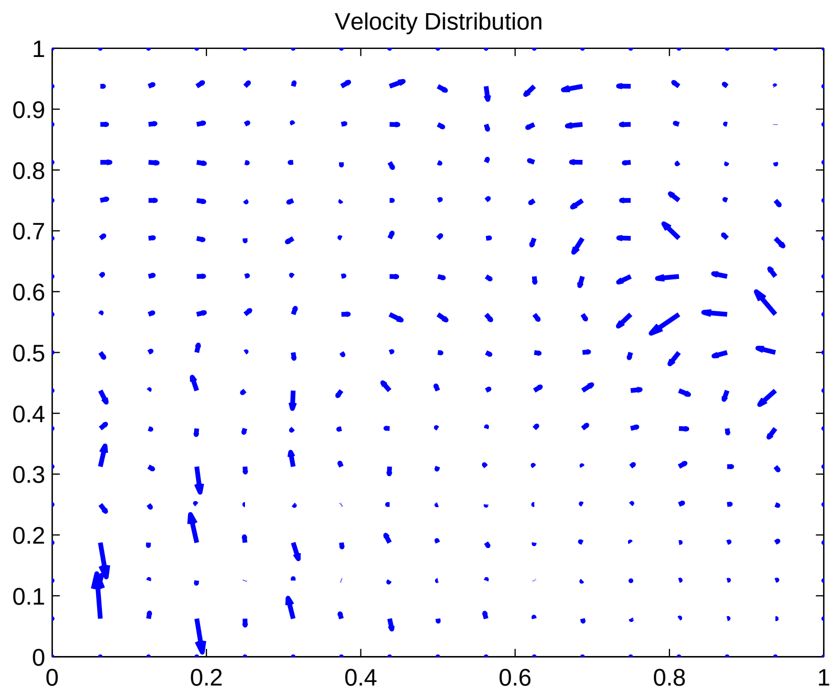

Figure 6.11: The velocity distribution for the randomly modified second realization of the permeability field.

Example 6.3:Using the third realization of the random permeability field in figure 6.1, we obtain the following results described in the next figures. Figure 6.12 shows the third realization of the permeability distribution.

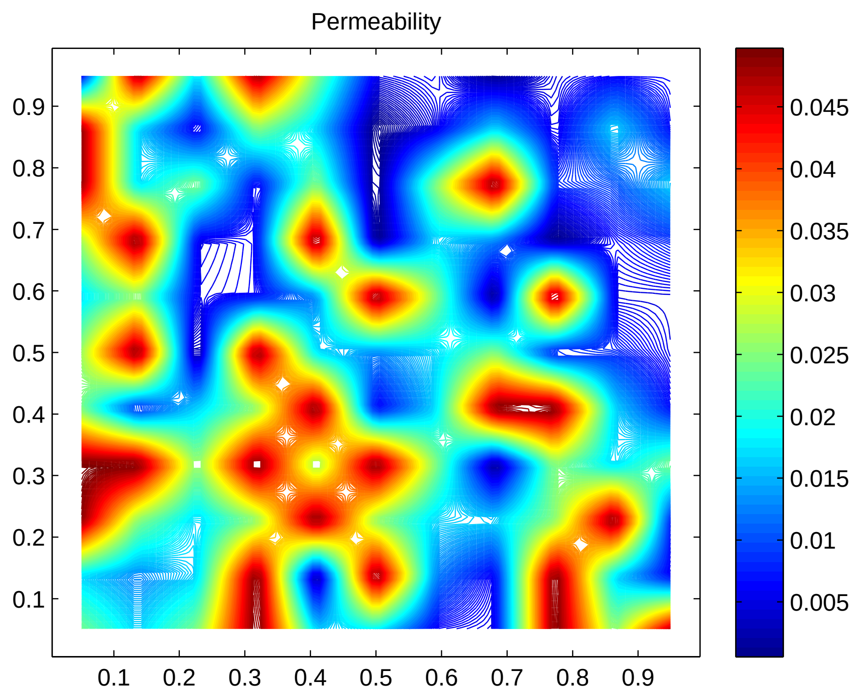

Figure 6.12: The third realization of the permeability distribution.

Figure 6.13 show the pressure distributions corresponding to this realization of the permeability field.

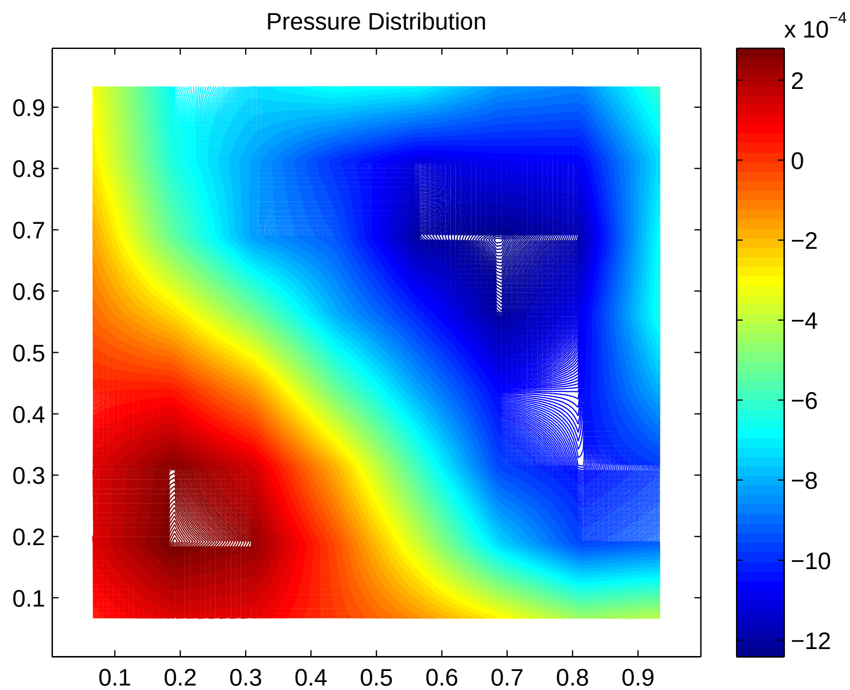

Figure 6.13: The pressure distributions for the third realization of the permeability field.

Figure 6.14 shows the pressure distribution corresponding to randomly modified permeability field from the third realization.

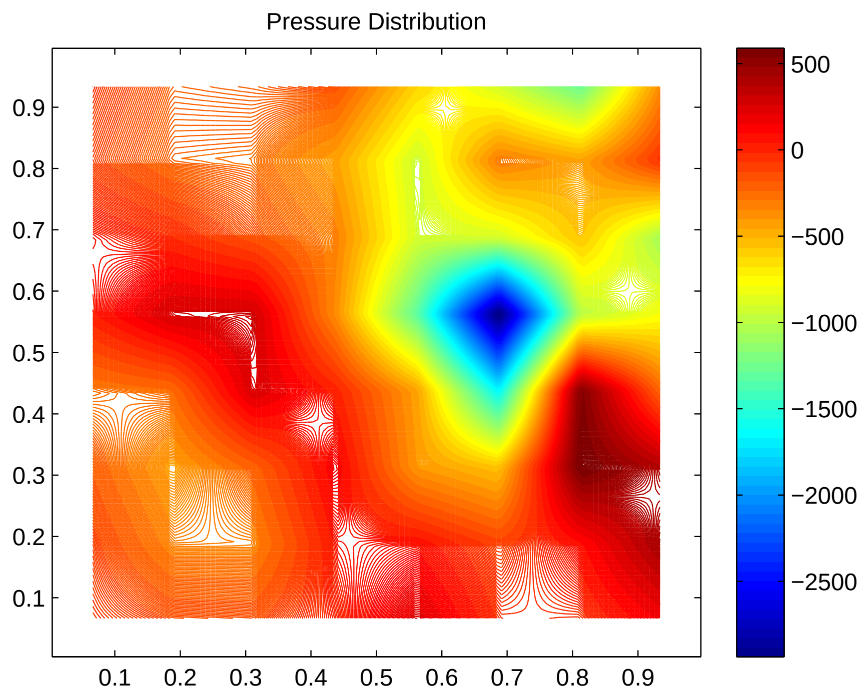

Figure 6.14: The pressure distribution of the corresponding to the randomly modified third realization of the permeability field.

Figure 6.15 shows that the pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generated random permeability field of the third realization.

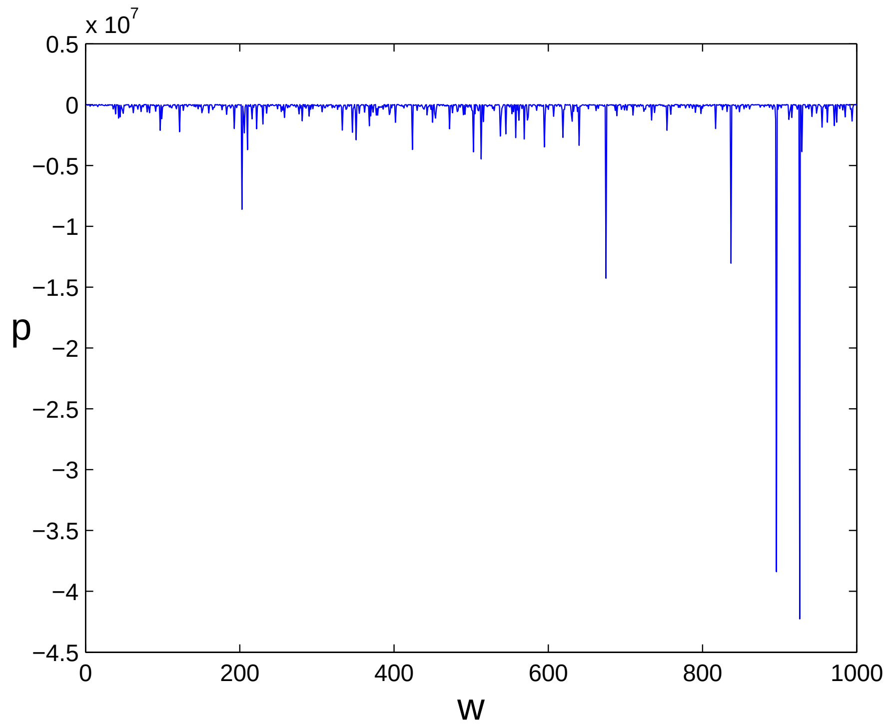

Figure 6.15: The pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generating random permeability field of the third realization.

Figure 6.16 shows the velocity distribution corresponding to the randomly modified third realization of the permeability field.

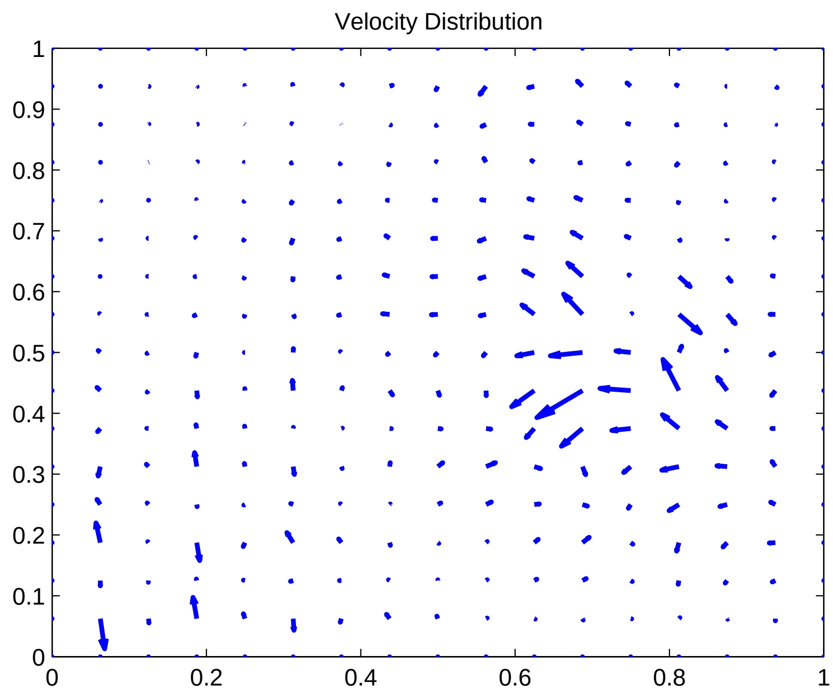

Figure 6.16: The velocity distribution for the randomly modified third realization of the permeability field.

Example 6.4:Using the fourth realization of the random permeability field in figure 6.1, we obtain the following results described in the next figures. Figure 6.17 shows the fourth realization of the permeability distribution.

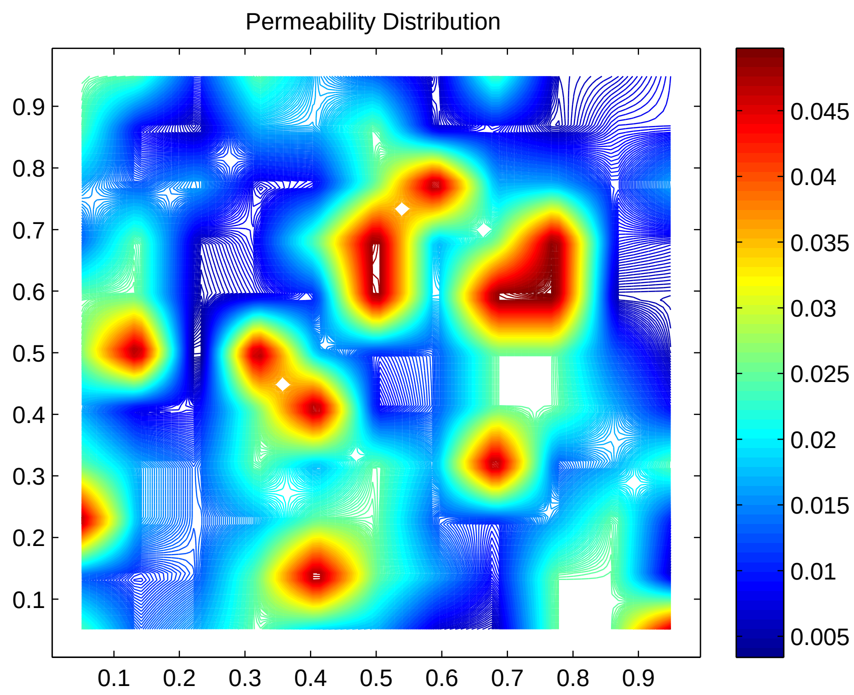

Figure 6.17: The fourth realization of the permeability distribution.

Figure 6.18 show the pressure distributions corresponding to this realization of the permeability field.

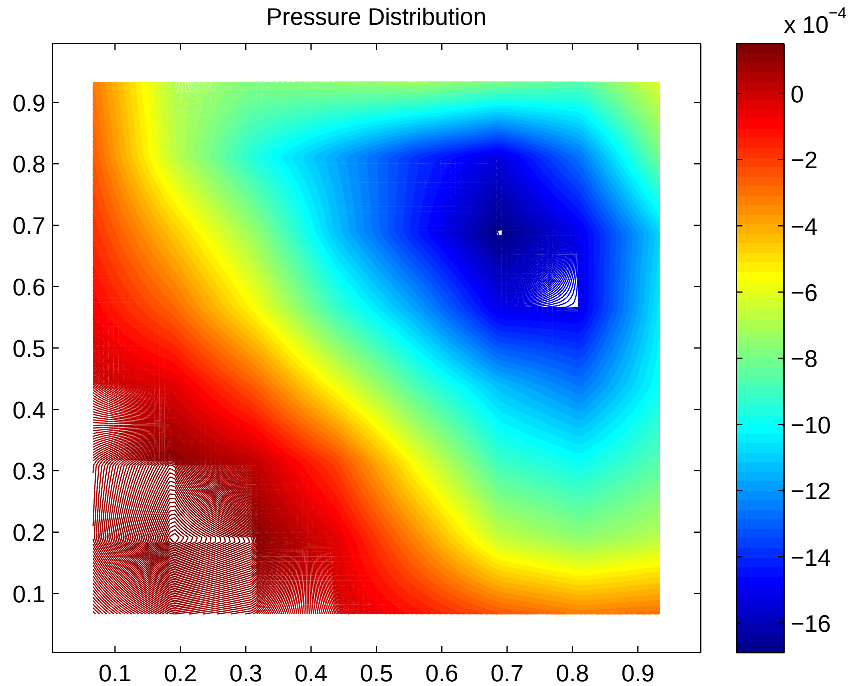

Figure 6.18: The pressure distributions for the fourth realization of the permeability field.

Figure 6.19 shows the pressure distribution corresponding to randomly modified permeability field from the fourth realization.

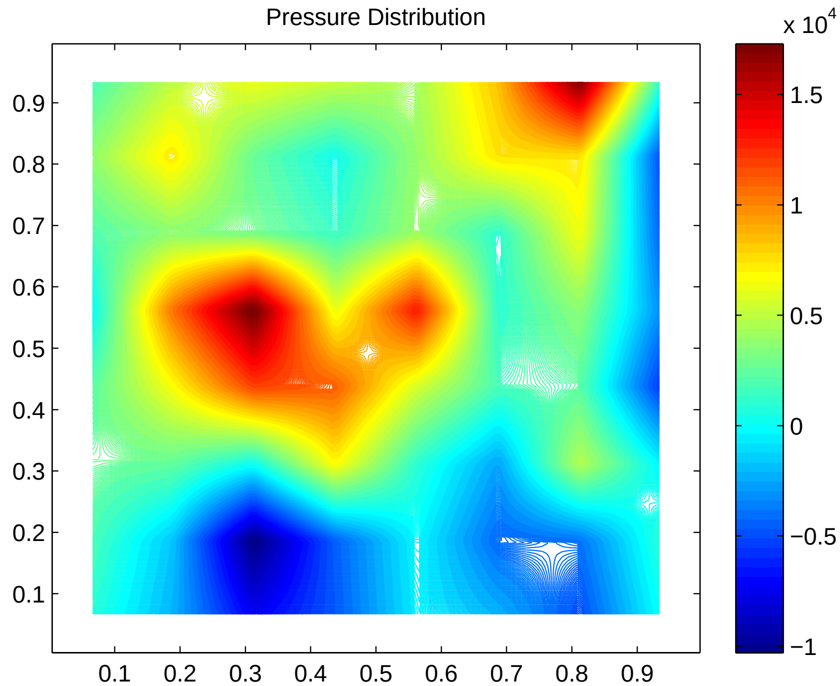

Figure 6.19: The pressure distribution corresponding to the randomly modified third realization of the permeability field.

Figure 6.20 shows that the pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generated random permeability field of the fourth realization.

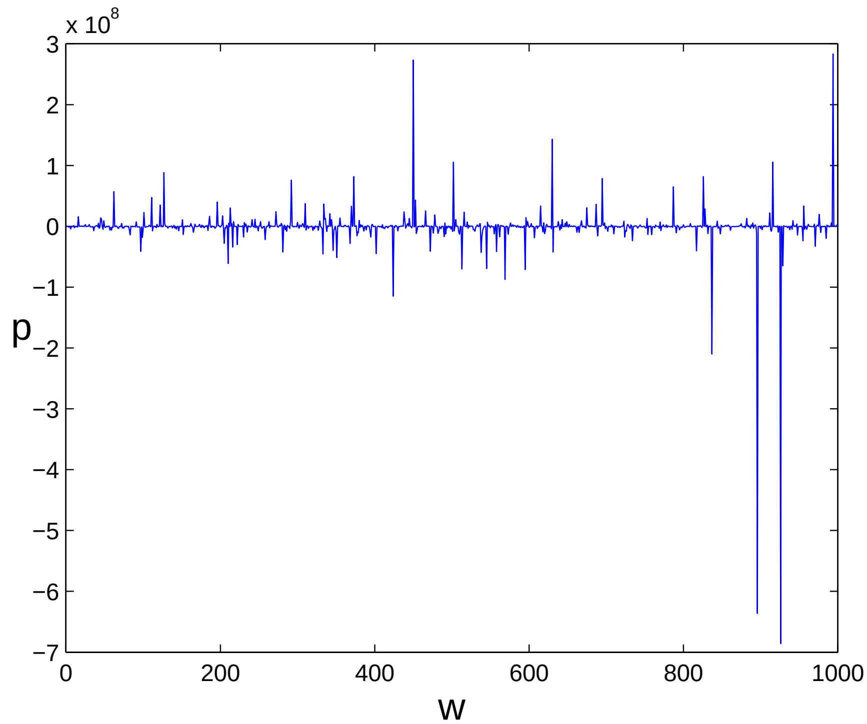

Figure 6.20: The pressure at $(x, y) = (0.5, 0.5)$ corresponding to the generating random permeability field of the third realization.

Figure 6.21 shows the velocity distribution corresponding to the randomly modified fourth realization of the permeability field.

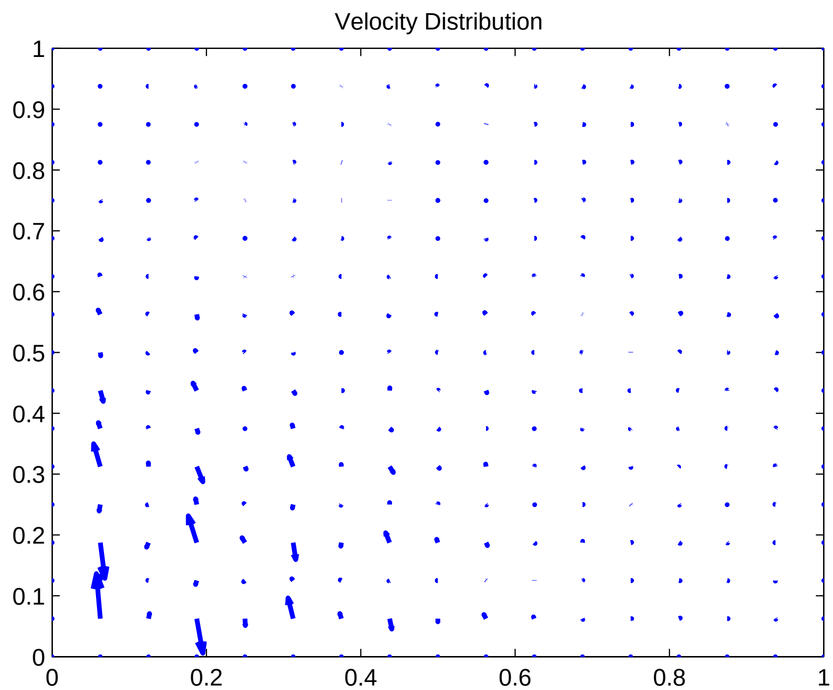

Figure 6.21: The velocity distribution for the randomly modified fourth realization of the permeability field.

Example 6.5: In this example we consider the permeability is taken as in the above four examples for all $x \in D = [0,1]^2$. We place water injection well at the center and production wells at the corners. We consider the neumann boundary condition. Figure 6.22 shows pressure contours for a homogeneous quarter-five spot corresponding to the first realization of the random permeability field.

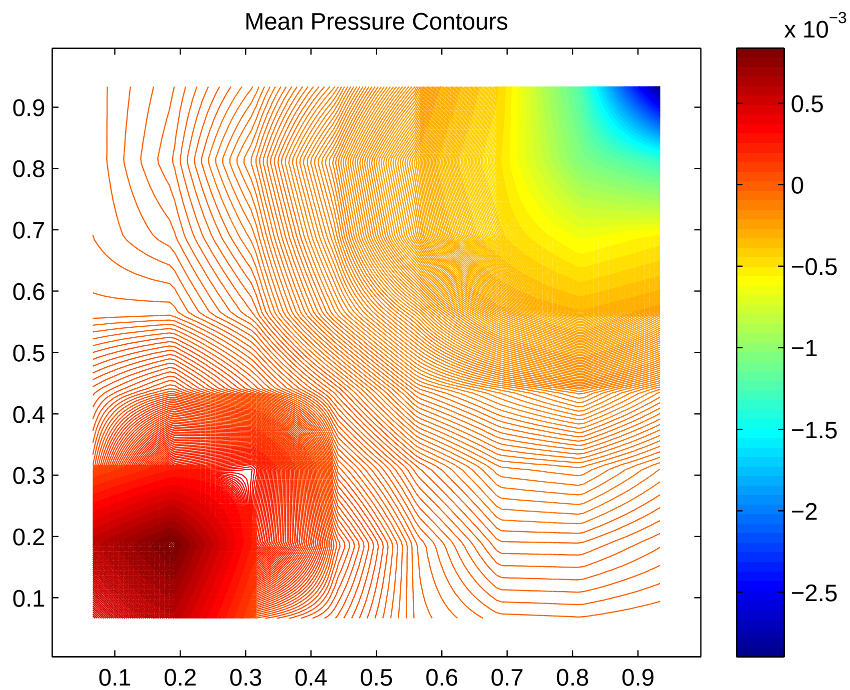

Figure 6.22: shows pressure contours corresponding to the first realization of the random permeability field.

Figure 6.23 shows pressure contours corresponding to the second realization of the random permeability field.

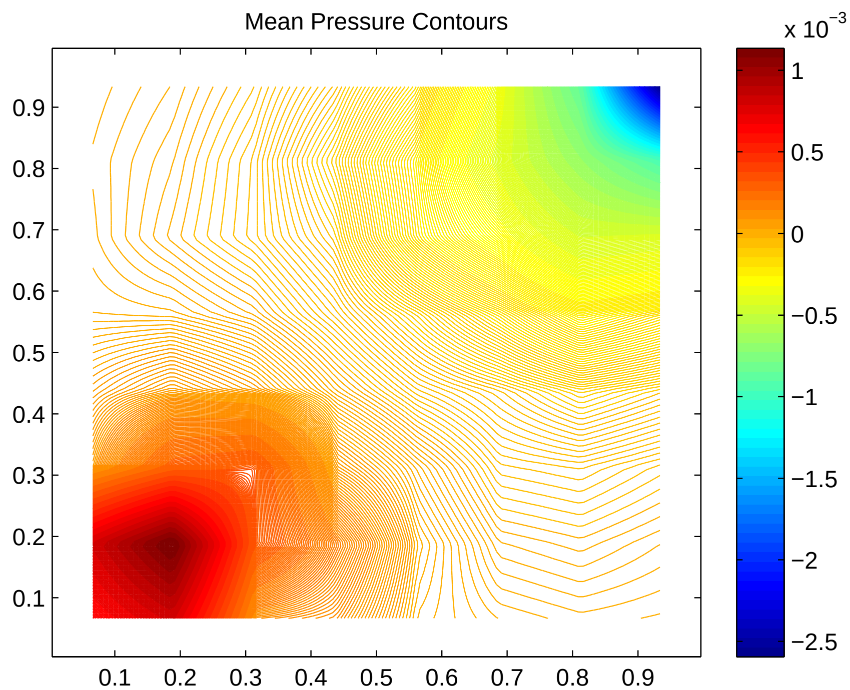

Figure 6.23: shows pressure contours corresponding to the second realization of the random permeability field.

Figure 6.24 shows pressure contours corresponding to the third realization of the random permeability field.

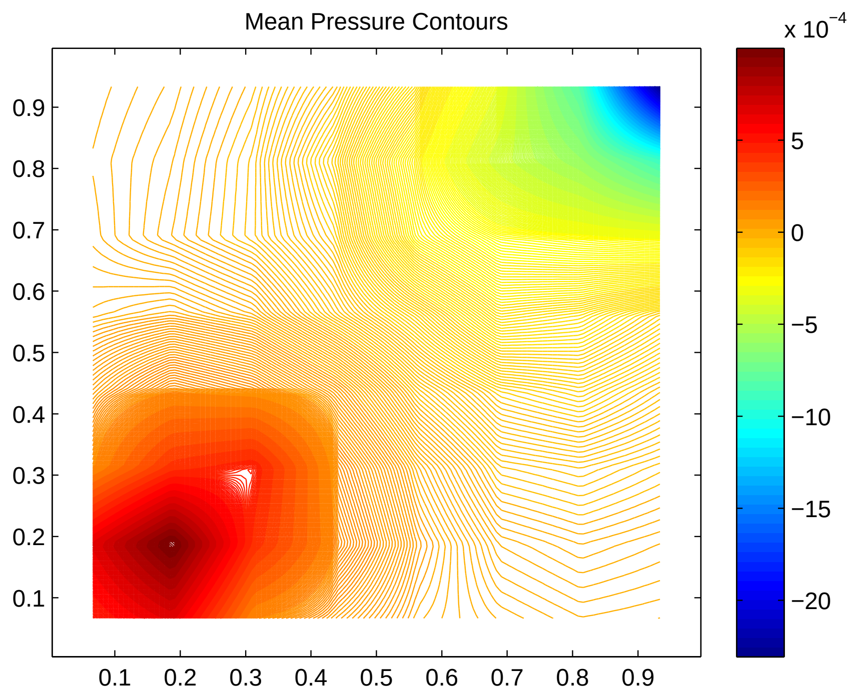

Figure 6.24: shows pressure contours corresponding to the third realization of the random permeability field.

Figure 6.25 shows pressure contours corresponding to the fourth realization of the random permeability field.

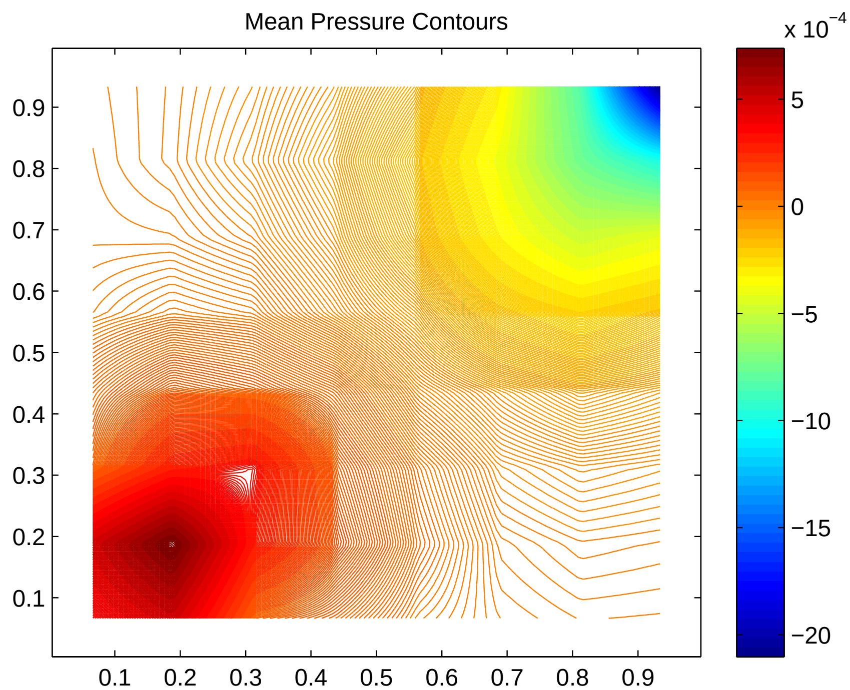

Figure 6.25: shows pressure contours corresponding to the fourth realization of the random permeability field.

CHAPTER 7

CONCLUSION AND FUTURE WORK

In this dissertation, the stochastic Darcy's equation with stochastic right-hand side has been studied. This equation was numerically solved by using SGFEM. We combined mixed finite element in computational domain with a stochastic basis function introduced from a subspace of $L^2(\Omega, \mathcal{F}, P)$ of martingale subspaces in the probability space. The advantage of this technique is reduced computation because of the exact computation of the stochastic integrals. We established existence, uniqueness, stability and the order of convergence. All results are introduced with real permeability data. We observed that high pressure and high velocity with high permeability as expected.

Corresponding to this work opens many possible directions organized as follow:

1. The Darcy's equation can be studied by using a different basis to represent the stochastic part.

2. The Darcy's equation can be studied by using wavelets instead of scaling functions. This has additional numerical advantages.

3. Study the extension to three dimensions.

4. The model can be studied by using n Brownian motion.

5. Extension to reservoir size. A typical size of a reservoir is $10m \times 10m \times 10m$ large and the dimension of the discretized problem becomes in the order of millions (upscaling).

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